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1. Parts (a) and (b) are separate problems with no relation.

   (a) (6 points) Determine if the following vectors are coplanar.

$$\begin{aligned}\mathbf{a} &= 2\mathbf{j} - 3\mathbf{k}\\ \mathbf{b} &= -3\mathbf{i} - 2\mathbf{j} + \mathbf{k}\\ \mathbf{c} &= \mathbf{i} - \mathbf{k}\end{aligned}$$

   (b) (8 points) A force of $\mathbf{F} = \langle 2, -2, 1\rangle$ is used to move a mass from $(1, -1, 2)$ to $(2, -1, 3)$. Assume the components of $\mathbf{F}$ are measured in Newtons, and that the units of the components of the points is meters. Find the work done in moving this mass. (Include the correct units in your final answer.)

   Solution:

   (a) Our answer will be in the affirmative if and only if the scalar triple product is 0.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 0 & 2 & -3\\ -3 & -2 & 1\\ 1 & 0 & -1\end{vmatrix} = 0(2-0) - 2(3-1) + (-3)(0-(-2)) = -10.$$

   So, we see that the three vectors are **not coplanar.**

   (b) We have displacement

$$\mathbf{D} = \langle 2-1, -1-(-1), 3-2\rangle = \langle 1, 0, 1\rangle.$$

   So, the work done is

$$W = \mathbf{F}\cdot\mathbf{D} = 2(1) - 2(0) + 1(1) = 3\text{N}\cdot\text{m or 3 joules.}$$

2. Consider the following two planes and line:

   - First Plane: $6x - 5y + z = 3$
   - Second Plane: $2x + y - 2z = 1$
   - Line: $\mathbf{r}(t) = \langle 1, 2, 0\rangle + t\langle 6, -5, 1\rangle$.

   (a) (8 points) Find the equation of the plane that contains the given **line** and the point $(0, 0, 1)$. Your final answer should be in the form $ax + by + cz = d$.

   (b) (8 points) Determine the angle between the **plane you found in (a)** and the **First Plane**.

   (c) (8 points) Find the parametric equations of the line of intersection of the **First Plane** and the **Second Plane**.

   Solution:

   (a) We need a point, which is given, and a normal vector. We can take the cross product of two non-parallel direction vectors of the plane to find a normal vector. One of these direction vectors can be given by the line from (a). The other can be found using the two points that have been given to find $\langle 1, 2, -1\rangle$. So, a normal vector is

$$\langle 6,-5,1\rangle \times \langle 1,2,-1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -5 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \langle 3,7,17\rangle.$$

Thus, the plane is

$$\begin{aligned} \langle 3,7,17\rangle \cdot \langle x,y,z-1\rangle &= 0 \\ 3x+7y+17z &= 17. \end{aligned}$$

(b) We need only find the angle between the normal vectors, which are $\mathbf{n}_a = \langle 3,7,17\rangle$ and $\mathbf{n}_1 = \langle 6,-5,1\rangle$, respectively. Thus, the angle between the planes is

$$\begin{aligned} \theta &= \arccos\left(\frac{\mathbf{n}_a \cdot \mathbf{n}_1}{||\mathbf{n}_a||||\mathbf{n}_1||}\right) \\ &= \arccos(0) \\ &= \frac{\pi}{2} \text{ radians.} \end{aligned}$$

(c) We need a point on this line and a direction vector. We begin by finding a point. There are infinitely many choices, but we will assume that $z = 0$ and find the point that satisfies this condition. Such a point will be a solution of this system:

$$\begin{aligned} 6x-5y &= 3 \\ 2x+y &= 1. \end{aligned}$$

This system has solution $x = \frac{1}{2}$ and $y = 0$, so we have point $\left(\frac{1}{2},0,0\right)$.
Since the direction vector of the line must be orthogonal to the normal vectors of both planes, then we can use the cross product of the two normal vectors to determine our direction vector:

$$\langle 6,-5,1\rangle \times \langle 2,1,-2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -5 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \langle 9,14,16\rangle.$$

Thus, the line is given by

$$\mathbf{r}(t) = \left\langle \frac{1}{2},0,0 \right\rangle + t\langle 9,14,16\rangle.$$

The parametric equations for this line are

$$\begin{aligned} x &= \frac{1}{2} + 9t \\ y &= 14t \\ z &= 16t. \end{aligned}$$

3. In this problem, we consider two quadric surfaces. The two quadric surfaces are not related.

(a) (10 points) Consider the quadric surface with points $P(x,y,z)$, for which the distance from P to the closest point on the z-axis is twice the distance from P to the closest point on the xy-plane

i. Determine the standard equation of the surface.
ii. Identity the type of surface.

(b) (16 points) Consider the quadric surface given by

$$x^2 - 2y^2 - 4z^2 - 4x + 24z - 36 = 0.$$

i. Determine the standard equation of the surface.
ii. Identity the type of surface.
iii. Determine the point(s) on the surface that intersects the line

$$\mathbf{r}(t) = \langle 10t + 2, 4\sqrt{2}t, 3t\rangle.$$

Solution:

(a) i. The closest point on the z-axis to (x, y, z) is $(0, 0, z)$, and the closest point to (x, y, z) on the xy-plane is $(x, y, 0)$. So, the surface's description translates to the following equation:

$$\sqrt{(x-0)^2 + (y-0)^2 + (z-z)^2} = 2\sqrt{(x-x)^2 + (y-y)^2 + (z-0)^2}.$$

We will square both sides and simplify in order to find a standard form:

$$x^2 + y^2 = 4z^2 \qquad \text{or} \qquad \frac{x^2}{4} + \frac{y^2}{4} = z^2.$$

ii. This surface is a **Cone**.

(b) i. We will complete the square and simplify to obtain a standard form:

$$\begin{aligned} x^2 - 2y^2 - 4z^2 - 4x + 24z - 36 &= 0 \\ (x^2 - 4x) - 2y^2 - 4(z^2 - 6z) &= 36 \\ (x^2 - 4x + 4) - 2y^2 - 4(z^2 - 6z + 9) &= 36 + 4 - 4(9) \\ (x-2)^2 - 2y^2 - 4(z-3)^2 &= 4 \\ \frac{(x-2)^2}{4} - \frac{y^2}{2} - (z-3)^2 &= 1. \end{aligned}$$

ii. This surface is a **Hyperboloid of Two Sheets.**

If we plug the parametric equations for this line into the form of the equation of the surface we obtained in (a), we have

$$\begin{aligned} \frac{((10t+2)-2)^2}{4} - \frac{(4\sqrt{2}t)^2}{2} - (3t-3)^2 &= 1 \\ 25t^2 - 16t^2 - (9t^2 - 18t + 9) &= 1 \\ 18t - 9 &= 1 \\ t &= \frac{5}{9}. \end{aligned}$$

We see that

$$\mathbf{r}\left(\frac{5}{9}\right) = \left\langle \frac{68}{9}, \frac{20\sqrt{2}}{9}, \frac{5}{3} \right\rangle.$$

So, the point of intersection is $\left(\frac{68}{9}, \frac{20\sqrt{2}}{9}, \frac{5}{3}\right)$.

4. There are two particles moving through space whose positions after t seconds are given by the following, where each component is measured in meters:

   - First Particle: $\mathbf{r}_1(t) = t\mathbf{i} + (1-t)\mathbf{j} + (t-t^2)\mathbf{k}$
   - Second Particle: $\mathbf{r}_2(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{2}t^{3/2}\mathbf{k}$

   (a) (9 points) Determine the rate of change of speed for the **First Particle** at time $t = 2$. (Include the correct units in your final answer.)

   (b) (9 points) Determine the curvature of the **First Particle** at the point $(1, 0, 0)$. (Include the correct units in your final answer.)

   (c) (9 points) How far does the **Second Particle** travel along its trajectory from $t = 0$ seconds to $t = \frac{4}{9}$ seconds? (Include the correct units in your final answer.)

   (d) (9 points) Do the two particles collide? If so, at what time **and** at what point?

Solution:

(a) The velocity of the first particle is given by

$$\mathbf{r}_1'(t) = \langle 1, -1, 1-2t \rangle.$$

So, its speed is given by

$$v(t) = ||\mathbf{r}_1'(t)|| = \sqrt{1^1 + (-1)^2 + (1-2t)^2} = \sqrt{3 - 4t + 4t^2}.$$

The rate of change of speed is therefore given by

$$v'(t) = \frac{-4+8t}{2\sqrt{3-4t+4t^2}} = \frac{-2+4t}{\sqrt{3-4t+4t^2}},$$

so the rate of change of speed at $t = 2$ is

$$v'(2) = \frac{6}{\sqrt{11}} \text{ meters per second squared.}$$

(b) We note that $\mathbf{r}_1(t) = \langle 1, 0, 0 \rangle$ when $t = 1$. We have $\mathbf{r}_1'(1) = \langle 1, -1, -1 \rangle$. And, since $\mathbf{r}_1''(t) = \langle 0, 0, -2 \rangle$, then we have $\mathbf{r}_1''(1) = \langle 0, 0, -2 \rangle$.
We have

$$\mathbf{r}_1'(1) \times \mathbf{r}_1''(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 0 & 0 & -2 \end{vmatrix} = \langle 2, 2, 0 \rangle.$$

So, the curvature at this point is given by

$$\kappa(1) = \frac{||\mathbf{r}_1'(1) \times \mathbf{r}_1''(1)||}{||\mathbf{r}_1'(1)||^3} = \frac{2\sqrt{2}}{3\sqrt{3}} \frac{1}{\text{meters}}.$$

(c) We need to compute the arc length. We have $\mathbf{r}_2'(t) = \left\langle 1, 1, \frac{3}{\sqrt{2}}\sqrt{t} \right\rangle$, so

$$||\mathbf{r}_2'(t)|| = \sqrt{1^1 + 1^2 + \left(\frac{3}{\sqrt{2}}\sqrt{t}\right)^2} = \sqrt{2 + \frac{9}{2}t}.$$

The arc length is given by

$$L = \int_0^{4/9} \sqrt{2 + \frac{9}{2}t}\, dt.$$

If we apply the substitution $u = 2 + \frac{9}{2}t$, then we have

$$L = \frac{2}{9}\int_2^4 u^{1/2}\, du = \left[\frac{4}{27}u^{3/2}\right]_2^4 = \frac{4}{27}\left(8 - 2^{3/2}\right) \text{ meters.}$$

(d) We can equate the components. From the second components, we have $1 - t = t$ which yields $t = 1/2$ seconds. This clearly works for the first two components. If there is a time when the particles collide, it must be at $t = 1/2$. But for the third component we have

$$t - t^2 = 1/4 \quad \text{and} \quad \sqrt{2}t^{3/2} = 1/2$$

when $t = 1/2$. Since $1/4 \neq 1/2$, there is no solution and the particles **do not collide.**