1. (16 pts) The following problems are unrelated.

   (a) Give the full partial fraction decomposition for the rational function

$$\frac{1}{(x-2)(x^4-16)(x^2+x+5)^2}$$

   You do NOT need to solve for the unknown constants.

   (b) Assume that $a(x)$ and $b(x)$ are two continuous functions such that $0 \leq a(x) \leq b(x)$. If you say yes to the following questions, explain why and state any theorems you use. If you say no, give appropriate examples of functions $a(x)$ and $b(x)$.

      i. Suppose that $\int_1^\infty b(x)\,dx$ converges. Must $\int_1^\infty a(x)\,dx$ converge too?

      ii. Suppose instead that $\int_1^\infty b(x)\,dx$ diverges. Must $\int_1^\infty a(x)\,dx$ diverge too?

Solution:

(a)

$$\frac{A}{x+2} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{x^2+x+5} + \frac{Hx+I}{(x^2+x+5)^2}$$

(b) i. Yes; this is a consequence of the Direct Comparison Test.

   ii. No; notice that $\frac{1}{x^2} \leq \frac{1}{x}$ on $[1,\infty)$ and $\int_1^\infty \frac{1}{x}\,dx$ diverges, but $\int_1^\infty \frac{1}{x^2}\,dx$ converges.

2. (36 pts) Evaluate the following expressions.

   (a) $\displaystyle\int \frac{3x-2}{x^2-2x+1}\,dx$

   (b) $\displaystyle\int \frac{x^2}{(1-x^2)^{5/2}}\,dx$ (*hint*: recall that $\int \sec^2(x)\,dx = \tan(x)$).

   (c) $\displaystyle\int e^x \sin(kx)\,dx$, where k is a constant.

Solution:

(a) Using partial fraction decomposition, we have

$$\frac{3x-2}{x^2-2x+1} = \frac{3x-2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}.$$

Multiplying by x^2-2x+1 yields

$$3x-2 = A(x-1) + B = Ax + (B-A).$$

Comparing coefficients, we see that $A = 3$ and $-2 = B - A$. Plugging $A = 3$ into the latter equation yields $B = 1$. This allows us to rewrite the integral:

$$\begin{aligned}\int \frac{3x-2}{x^2-2x+1}\,dx &= \int \left(\frac{3}{x-1} + \frac{1}{(x-1)^2}\right)dx \\ &= \int \frac{3}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx \\ &= 3\ln|x-1| - \frac{1}{x-1} + C.\end{aligned}$$

(b) Using the substitution $x = \sin\theta$, $dx = \cos\theta\, d\theta$, we get

$$\begin{aligned}\int \frac{x^2}{(1-x^2)^{5/2}}\, dx &= \int \frac{\sin^2\theta}{(1-\sin^2\theta)^{5/2}} \cos\theta\, d\theta \\ &= \int \frac{\sin^2\theta\cos\theta}{\cos^5\theta}\, d\theta \\ &= \int \frac{\sin^2\theta}{\cos^4\theta}\, d\theta \\ &= \int \frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{1}{\cos^2\theta}\, d\theta \\ &= \int \tan^2\theta\sec^2\theta\, d\theta\end{aligned}$$

Using the u-substitution $u = \tan\theta$ and $du = \sec^2\theta$, we get

$$\begin{aligned}&= \int u^2\, du \\ &= \frac{1}{3}u^3 + C \\ &= \frac{1}{3}\tan^3\theta + C.\end{aligned}$$

From the reference triangle for $x = \sin\theta$, we know that

$$\tan\theta = \frac{x}{\sqrt{1-x^2}}\, dx.$$

Hence,

$$\int \frac{x^2}{(1-x^2)^{5/2}}\, dx = \frac{1}{3}\left(\frac{x}{\sqrt{1-x^2}}\right)^3 + C.$$

(c) This will require two applications of integration by parts. First, set $u = \sin(kx)$ and $dv = e^x dx$. Then $du = k\cos(kx)$ and $v = e^x$, so

$$\int e^x \sin(kx)\, dx = e^x\sin(kx) - k\int e^x\cos(kx)\, dx.$$

Applying IBP again with $u = \cos(kx)$, $du = -k\sin(kx)$, $dv = e^x dx$, and $v = e^x$ yields

$$\begin{aligned}\int e^x \sin(kx)\, dx &= e^x\sin(kx) - k\int e^x\cos(kx)\, dx \\ &= e^x\sin(kx) - k\left(e^x\cos(kx) + k\int e^x\sin(kx)\, dx\right) \\ &= e^x\sin(kx) - ke^x\cos(kx) - k^2\int e^x\sin(kx)\, dx\end{aligned}$$

Combining the like terms gives

$$\begin{aligned}\int e^x\sin(kx)\, dx + k^2\int e^x\sin(kx)\, dx &= e^x\sin(kx) - ke^x\cos(kx) \\ (1+k^2)\int e^x\sin(kx)\, dx &= e^x\sin(kx) - ke^x\cos(kx)\end{aligned}$$

Dividing through by $(1+k^2)$ gives the result:

$$\int e^x\sin(kx)\, dx = \frac{e^x\sin(kx) - ke^x\cos(kx)}{1+k^2} + C.$$

3. (24 pts) Evaluate each expression below. If an integral converges, determine its value. If an integral diverges, justify why.

   (a) $\displaystyle\int_0^\infty \frac{dt}{(1+t^2)(1+\arctan(t))^2}$

   (b) $\displaystyle\int_0^3 \frac{x-1}{x^{4/3}}\,dx$

   Solution:

   (a) Setting $u = 1+\arctan(t)$ and $du = \frac{1}{1+t^2}dt$, we get

$$\begin{aligned}\int_0^\infty \frac{dt}{(1+t^2)(1+\arctan(t))^2} &= \lim_{r\to\infty}\int_1^{1+\arctan(r)} \frac{1}{u^2}\,du\\ &= -\lim_{r\to\infty}\left.\frac{1}{u}\right|_1^{1+\arctan(r)}\\ &= -\lim_{r\to\infty}\left(\frac{1}{1+\arctan(r)}-1\right)\\ &= -\left(\frac{1}{1+\pi/2}-1\right) = 1-\frac{2}{2+\pi} = \frac{\pi}{2+\pi}.\end{aligned}$$

   This problem can also be done using the substitution $t = \tan(x)$.

   (b) This function has a discontinuity at $x = 0$, so we evaluate the limit

$$\begin{aligned}\lim_{t\to 0^+}\int_t^3 \frac{x-1}{x^{4/3}}\,dx &= \lim_{t\to 0^+}\int_t^3 \left(x^{-1/3}-x^{-4/3}\right)\,dx\\ &= \lim_{t\to 0^+}\left.\left(\frac{3}{2}x^{2/3}+3x^{-1/3}\right)\right|_t^3\\ &= \lim_{t\to 0^+}\left(\frac{3}{2}3^{3/2}+3\cdot 3^{-1/3}-\frac{3}{2}t^{3/2}-3t^{-1/3}\right)\\ &= \frac{3}{2}3^{3/2}-3\cdot 3^{-1/3}-\infty\\ &= -\infty\end{aligned}$$

   So this integral diverges.

4. (12 pts) The table below records the velocity $v(t)$ in meters per second of a car at time t (measured in seconds):

t	0	2	4	6	8	10	12
$v(t)$	0	5.1	10.9	16.8	22.8	28.9	35.1

   (a) Give an approximation for the distance the car travels over the first 12 seconds using the trapezoidal rule with $n = 3$ (You do NOT need to simplify your expression, but you should give an expression which could be easily entered into a calculator).

   (b) Sort the values L_3, R_3, and T_3 in order from least to greatest. *Remember that these are the left-sided, right-sided, and trapezoidal approximations with* $n = 3$.

   Solution:

(a) For $n = 3$, $\Delta t = (12 - 0)/3 = 4$. So the the trapezoidal approximation is

$$T_3 = \frac{4}{2} \cdot (0 + 2(10.9) + 2(22.8) + 35.1).$$

(b) Since the values in the table are increasing, $L_3 \leq R_3$. Since $T_3 = \frac{1}{2}(L_3 + R_3)$, we know that the approximations L_3, T_3, R_3 are ordered from least to greatest.

5. (12 pts) The function $f(x)$ has a second derivative given by

$$f''(x) = \frac{x\sin(x) - \cos^2(x)}{1 + x^3}.$$

Determine how large n should be so that the midpoint approximation M_n for the integral $\int_2^3 f(x)\,dx$ is within 10^{-3} of the true value. *You should solve for n, but you do not need to simplify your answer.*

Solution: First, we find an upper bound on $|f''(x)|$:

$$|f''(x)| \leq \left|\frac{x\sin(x) - \cos(x)}{1 + x^3}\right| \leq \frac{|x| \cdot |\sin(x)| + |\cos(x)|}{|1 + x^3|} \leq \frac{(3)(1) + 1}{1 + 8} = \frac{4}{9}.$$

We can estimate the error $|E_M|$ by

$$|E_M| \leq \frac{(4/9)(3-2)^3}{24n^2} = \frac{1}{54n^2} \leq 10^{-3}.$$

Rearranging the final inequality and solving for n, we find that choosing

$$n > \sqrt{\frac{1000}{54}}$$

will guarantee an error less than 10^{-3} for the Midpoint Rule.