1. [2360/020426 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a two-column table (letter - answer) completely separate from any work you do to arrive at the answer.

   (a) The differential equation $x' + 2xt - tx^2 = t$ has two equilibrium solutions.

   (b) $y = \sqrt{t}$ is a solution of $2t^2y'' - ty' + y = 0$ on the interval $t > 0$.

   (c) $y' = 2y^3 + 3y^2 - 1$ has a semistable equilibrium at $y = \frac{1}{2}$.

   (d) The general solution of $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2x}{\sin y + y\cos y}$ is given implicitly by $x^2 - y\sin y = 1$.

   (e) Let L be a linear operator and suppose that $\vec{\mathbf{x}}_1$ and $\vec{\mathbf{x}}_2$ are two solutions of $L(\vec{\mathbf{x}}) = f(t)$. Then $\vec{\mathbf{x}}_1 - \vec{\mathbf{x}}_2$ is a solution of the associated homogeneous problem.

   SOLUTION:

   (a) **FALSE** $x' = t - 2xt + tx^2 = t(1 - 2x + x^2) = t(1-x)^2$, implying that $x = 1$ is the only equilibrium solution.

   (b) **TRUE** Substituting $y = \sqrt{t}$ into the DE,

$$2t^2\left(\sqrt{t}\right)'' - t\left(\sqrt{t}\right)' + \sqrt{t} = 2t^2\left(-\frac{1}{4}t^{-3/2}\right) - t\left(\frac{1}{2}t^{-1/2}\right) + t^{1/2} = -\frac{1}{2}t^{1/2} - \frac{1}{2}t^{1/2} + t^{1/2} = 0$$

   (c) **FALSE** $2\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 - 1 = \frac{2}{8} + \frac{3}{4} - 1 = 0 \implies y = \frac{1}{2}$ is an equilibrium solution. If $y < \frac{1}{2}, y' < 0$ and if $y > \frac{1}{2}, y' > 0$ implying the equilibrium is unstable.

   (d) **FALSE** There are no parameters in $x^2 - y\sin y = 1$ and thus this cannot be the general solution. It is, however, a particular solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 - y\sin y = 1\right) \implies 2x - y\cos y\frac{\mathrm{d}y}{\mathrm{d}x} - \sin y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{\sin y + y\cos y}$$

   (e) **TRUE** $L\left(\vec{\mathbf{x}}_1 - \vec{\mathbf{x}}_2\right) = L\left(\vec{\mathbf{x}}_1\right) - L\left(\vec{\mathbf{x}}_2\right) = f(t) - f(t) = 0$

■

2. [2360/020426 (15 pts)] You recently inherited \$16,094.38 which you will use to help pay off some student loans. Your local bank is offering accounts earning 4% interest compounded continuously. The loan repayment will require you to withdraw money from the account at a rate of $e^{t/25}/(t+2)$ (tens of thousands of dollars per year). Let $A(t)$ be the amount of money (tens of thousands of dollars) in the account at time $t \geq 0$, where t is measured in years. The scenario can be modeled by the initial value problem

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{A}{25} - \frac{e^{t/25}}{t+2},\ A(0) = \ln 5$$

   Will the bank account run out of money in a finite amount of time? If so, find that time. If not, explain why not. Use the Euler-Lagrange Two Stage Method (variation of parameters) to answer the question. Simply plugging into formulas, not showing all relevant steps or using another method will result in zero credit.

   SOLUTION:

   Solve the homogeneous problem using separation of variables.

$$\int \frac{\mathrm{d}A_h}{A_h} = \frac{1}{25}\int \mathrm{d}t$$

$$\ln|A_h| = \frac{t}{25} + k$$

$$A_h = Ce^{t/25}$$

   Now let $A_p = v(t)e^{t/25}$. Substituting this into the nonhomogeneous equation yields

$$A_p' + \frac{A_p}{25} = \frac{v}{25}e^{t/25} + v'e^{t/25} - \frac{v}{25}e^{t/25} = v'e^{t/25} = -\frac{e^{t/25}}{t+2}$$

$$v' = -\frac{1}{t+2}$$

$$v = \int v'\,\mathrm{d}t = -\int \frac{\mathrm{d}t}{t+2} = -\ln|t+2| = -\ln(t+2) \qquad \text{since } t \geq 0$$

$$A_p = -e^{t/25}\ln(t+2)$$

Thus $A(t) = A_h + A_p = e^{t/25}\left[C - \ln(t+2)\right]$. Applying the initial condition gives

$$A(0) = \ln 5 = C - \ln 2 \implies C = \ln 5 + \ln 2 = \ln 10 \implies A(t) = e^{t/25} \ln \frac{10}{t+2}$$

The bank account will be devoid of funds if $A(t) = 0$. Since $e^{t/25} > 0$ for all t, we must have

$$\ln \frac{10}{t+2} = 0 \implies \frac{10}{t+2} = 1 \implies t = 8$$

The account will contain no money after 8 years. ■

3. [2360/020426 (15 pts)] Consider the modified form of Newton's Law of Cooling given by $T'(t) = (\sin t)(T-5),\ T(0) = T_0$ where $T(t)$ is the temperature of an object.

   (a) (3 pts) If $T_0 = 5$, find $T(t)$, justifying your answer in words only without any calculations.

   (b) (12 pts) If $T_0 = 10$, find the first time after $t = 0$ when the object's temperature returns to T_0. You must use separation of variables to solve this problem. Zero credit for using another method or not showing all relevant steps.

SOLUTION:

(a) Since $T = 5$ is the equilibrium solution, the temperature of the object never changes, that is, $T(t) = 5$.

(b)

$$\begin{aligned}
\int \frac{\mathrm{d}T}{T-5} &= \int \sin t \,\mathrm{d}t \\
\ln|T-5| &= -\cos t + k \\
|T-5| &= e^k e^{-\cos t} \\
T - 5 &= Ce^{-\cos t} \qquad \text{apply initial condition} \\
10 - 5 &= Ce^{-1} \implies C = 5e \\
T(t) &= 5\left(1 + e^{1-\cos t}\right)
\end{aligned}$$

The object will reach the initial temperature again if

$$\begin{aligned}
10 &= 5(1 + e^{1-\cos t}) \\
2 &= 1 + e^{1-\cos t} \\
1 &= e^{1-\cos t} \\
\ln 1 &= (1 - \cos t)\ln e \\
-1 &= -\cos t \implies \cos t = 1 \implies t = 2\pi
\end{aligned}$$

■

4. [2360/020426 (12 pts)] A 1000 gallon pot is initially 80 percent full of sweet tea in which 100 ounces of sugar is dissolved. Tea containing $1/(t+1)$ ounces of sugar per gallon enters the pot at 5 gallons per minute. The well-mixed sweet tea leaves the pot at 7 gallons per minute.

   (a) (10 pts) Set up, but **DO NOT SOLVE**, an initial value problem for the amount of sugar, S, contained in the pot at time t.

   (b) (2 pts) If the initial time is $t = 0$, over what interval will the solution be valid? You do not need to find the solution to the IVP to answer this question.

SOLUTION:

(a) Since the flow rates differ, the volume of sweet tea in the pot will vary with time. To determine this,

$$\frac{dV}{dt} = \text{flow rate in} - \text{flow rate out} = 5 - 7 = -2,\ V(0) = 800$$

$$\int dV = \int -2\,dt$$

$$V(t) = -2t + C$$

$$V(0) = 800 = 2(0) + C$$

$$V(t) = -2t + 800$$

$$\frac{dS}{dt} = \text{mass rate in} - \text{mass rate out} = \left(\frac{1}{t+1}\frac{\text{ounce}}{\text{gallon}}\right)\left(5\frac{\text{gallon}}{\text{minute}}\right) - \left(\frac{S}{-2t+800}\frac{\text{ounce}}{\text{gallon}}\right)\left(7\frac{\text{gallon}}{\text{minute}}\right)$$

$$\frac{dS}{dt} + \frac{7S}{800-2t} = \frac{5}{t+1},\ S(0) = 100$$

(b) The tank will be empty when $t = 400$, so the interval over which the solution of the differential equation is valid is $[0, 400]$.

■

5. [2360/020426 (24 pts)] Consider the initial value problem $y' + \frac{y}{t} = y^2,\ y(1) = \frac{1}{3}$ with $t > 0$.

(a) (6 pts) What conclusions, if any, can be drawn from Picard's theorem regarding solutions to the initial value problem? Justify your answer.

(b) (6 pts) Approximate $y(1.5)$ using one step of Euler's Method.

(c) (12 pts) Solve the initial value problem by first dividing the differential equation by y^2, using the substitution $v = y^{-1}$ and then the integrating factor method. Simply plugging into formulas, not showing all relevant steps or using another method will result in zero credit. Note: this is a Bernoulli differential equation which was studied in the homework.

SOLUTION:

(a) Rewrite the equation as $y' = y^2 - \frac{y}{t}$. Then $f(t,y) = y^2 - \frac{y}{t}$ and $f_y(t,y) = 2y - \frac{1}{t}$. A rectangle containing $\left(1, \frac{1}{3}\right)$ can be found wherein both f and f_y are continuous, for example, any rectangle that does not contain the y-axis ($t = 0$). Then Picard's theorem guarantees that a unique solution to the initial value problem exists on some open interval around $t = 1$.

(b) Euler's method is $y_{i+1} = y_i + h\left(y_i^2 - \frac{y_i}{t_i}\right)$. Using one step to approximate $y(1.5)$ requires $h = \frac{1}{2}$ and

$$y(1.5) \approx y_1 = y_0 + h\left(y_0^2 - \frac{y_0}{t_0}\right) = \frac{1}{3} + \frac{1}{2}\left[\left(\frac{1}{3}\right)^2 - \frac{1/3}{1}\right] = \frac{1}{3} + \frac{1}{2}\left(-\frac{2}{9}\right) = \frac{2}{9}$$

(c) The substitution $v = y^{-1} \implies v' = -y^{-2}y'$. Dividing the differential equation by y^2 and using the substitution

$$y^{-2}y' + \frac{y^{-1}}{t} = 1$$

$$-v' + \frac{v}{t} = 1 \implies v' - \frac{v}{t} = -1$$

which is a linear equation. Noting that $t > 0$, the integrating factor is $\mu(t) = e^{\int -dt/t} = \frac{1}{t}$. Then

$$\int \left(\frac{v}{t}\right)' dt = -\int \frac{dt}{t}$$

$$\frac{v}{t} = -\ln t + C \implies v = Ct - t\ln t \implies y = \frac{1}{Ct - t\ln t}$$

Applying the initial condition $\frac{1}{3} = \frac{1}{C} \implies C = 3$ giving $y = \frac{1}{3t - t\ln t}$ as the solution to the IVP.

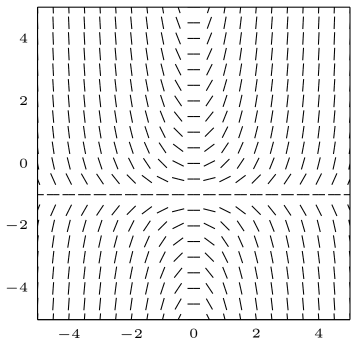
■

6. [2360/020426 (12 pts)] Write the letters a. through f. in a single column in your bluebook. Next to each letter, write the Roman numeral (or the word NONE) of the direction field corresponding to the given differential equation. No partial credit is available and no work need be shown.

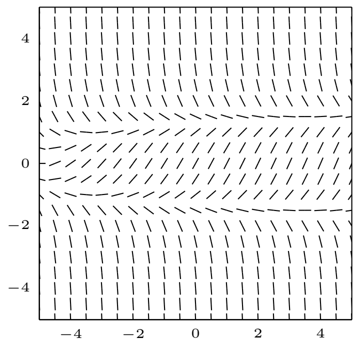
(a) $y' = y^2 + t$ (b) $y' = 2y - y^2$ (c) $y' + y^2 = \ln(t+6)$

(d) $y' = ty + t$ (e) $y' = \sin 2y \cos 2t$ (f) $y' - \sin 2t \cos 2y = 0$

I



II



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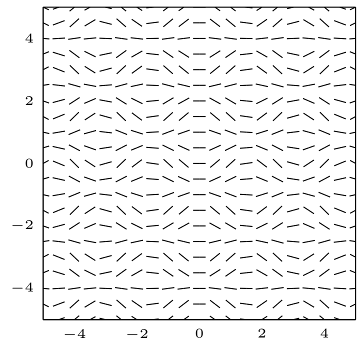
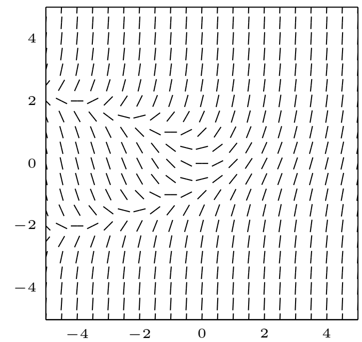


IV



SOLUTION:

(a) IV

(b) NONE

(c) II

(d) I

(e) NONE

(f) III

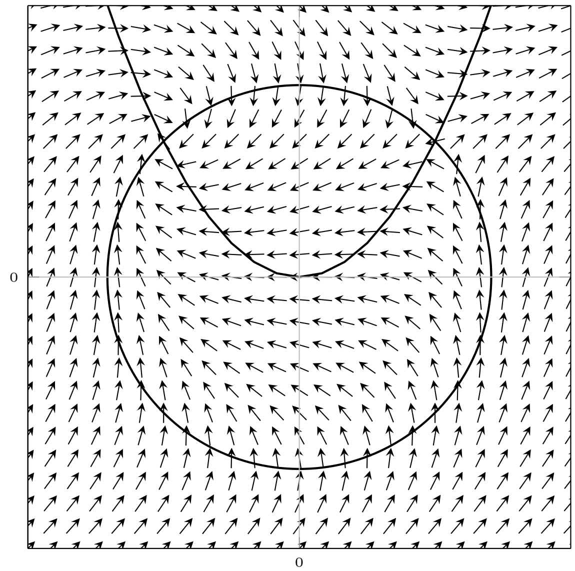
7. [2360/020426 (12 pts)] Consider the system of differential equations: $\begin{aligned} x' &= x^2 + y^2 - 2 \\ y' &= x^2 - y \end{aligned}$. The phase plane for this system is shown below with the origin, $(0,0)$, at the center of the figure.

(a) (2 pts) With regard to the system of differential equations, what does the circle represent?

(b) (2 pts) With regard to the system of differential equations, what does the parabola represent?

(c) (6 pts) Find all equilibrium solutions and determine their stability.

(d) (2 pts) If the initial condition is $(x(0), y(0)) = (0,0)$ find: i. $\lim_{t\to\infty} x(t)$ ii. $\lim_{t\to\infty} y(t)$



SOLUTION:

(a) v nullcline

(b) h nullcline

(c) These occur at the intersections of the nullclines, solutions of the following nonlinear system.

$$x^2 + y^2 - 2 = 0 \tag{1}$$

$$x^2 - y = 0 \implies y = x^2 \tag{2}$$

Using (2) in (1) gives

$$y^2 + y - 2 = (y + 2)(y - 1) = 0 \implies y = -2, 1$$

If $y = -2$, (2) has no real solution. If $y = 1$ in (2), then $x = \pm 1$. The equilibrium solutions are $(-1, 1)$ and $(1, 1)$. From the vector field, $(-1, 1)$ is stable and $(1, 1)$ is unstable.

(d) Again using the vector field and tracing the trajectory from the origin, $\lim_{t\to\infty} x(t) = -1$ and $\lim_{t\to\infty} y(t) = 1$.

■