- This exam is worth 100 points and has 7 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"$\times$ 11" crib sheet with writing on one side.

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0. At the top of the page containing your solution to problem 1, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/020426 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a two-column table (letter - answer) completely separate from any work you do to arrive at the answer.

   (a) The differential equation $x' + 2xt - tx^2 = t$ has two equilibrium solutions.

   (b) $y = \sqrt{t}$ is a solution of $2t^2y'' - ty' + y = 0$ on the interval $t > 0$.

   (c) $y' = 2y^3 + 3y^2 - 1$ has a semistable equilibrium at $y = \frac{1}{2}$.

   (d) The general solution of $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2x}{\sin y + y\cos y}$ is given implicitly by $x^2 - y\sin y = 1$.

   (e) Let L be a linear operator and suppose that $\vec{\mathbf{x}}_1$ and $\vec{\mathbf{x}}_2$ are two solutions of $L(\vec{\mathbf{x}}) = f(t)$. Then $\vec{\mathbf{x}}_1 - \vec{\mathbf{x}}_2$ is a solution of the associated homogeneous problem.

2. [2360/020426 (15 pts)] You recently inherited \$16,094.38 which you will use to help pay off some student loans. Your local bank is offering accounts earning 4% interest compounded continuously. The loan repayment will require you to withdraw money from the account at a rate of $e^{t/25}/(t+2)$ (tens of thousands of dollars per year). Let $A(t)$ be the amount of money (tens of thousands of dollars) in the account at time $t \geq 0$, where t is measured in years. The scenario can be modeled by the initial value problem

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{A}{25} - \frac{e^{t/25}}{t+2},\ A(0) = \ln 5$$

   Will the bank account run out of money in a finite amount of time? If so, find that time. If not, explain why not. Use the Euler-Lagrange Two Stage Method (variation of parameters) to answer the question. Simply plugging into formulas, not showing all relevant steps or using another method will result in zero credit.

3. [2360/020426 (15 pts)] Consider the modified form of Newton's Law of Cooling given by $T'(t) = (\sin t)(T-5),\ T(0) = T_0$ where $T(t)$ is the temperature of an object.

   (a) (3 pts) If $T_0 = 5$, find $T(t)$, justifying your answer in words only without any calculations.

   (b) (12 pts) If $T_0 = 10$, find the first time after $t = 0$ when the object's temperature returns to T_0. You must use separation of variables to solve this problem. Zero credit for using another method or not showing all relevant steps.

4. [2360/020426 (12 pts)] A 1000 gallon pot is initially 80 percent full of sweet tea in which 100 ounces of sugar is dissolved. Tea containing $1/(t+1)$ ounces of sugar per gallon enters the pot at 5 gallons per minute. The well-mixed sweet tea leaves the pot at 7 gallons per minute.

   (a) (10 pts) Set up, but **DO NOT SOLVE**, an initial value problem for the amount of sugar, S, contained in the pot at time t.

   (b) (2 pts) If the initial time is $t = 0$, over what interval will the solution be valid? You do not need to find the solution to the IVP to answer this question.

MORE PROBLEMS BELOW/ON REVERSE

5. [2360/020426 (24 pts)] Consider the initial value problem $y' + \dfrac{y}{t} = y^2,\ y(1) = \dfrac{1}{3}$ with $t > 0$.

   (a) (6 pts) What conclusions, if any, can be drawn from Picard's theorem regarding solutions to the initial value problem? Justify your answer.

   (b) (6 pts) Approximate $y(1.5)$ using one step of Euler's Method.

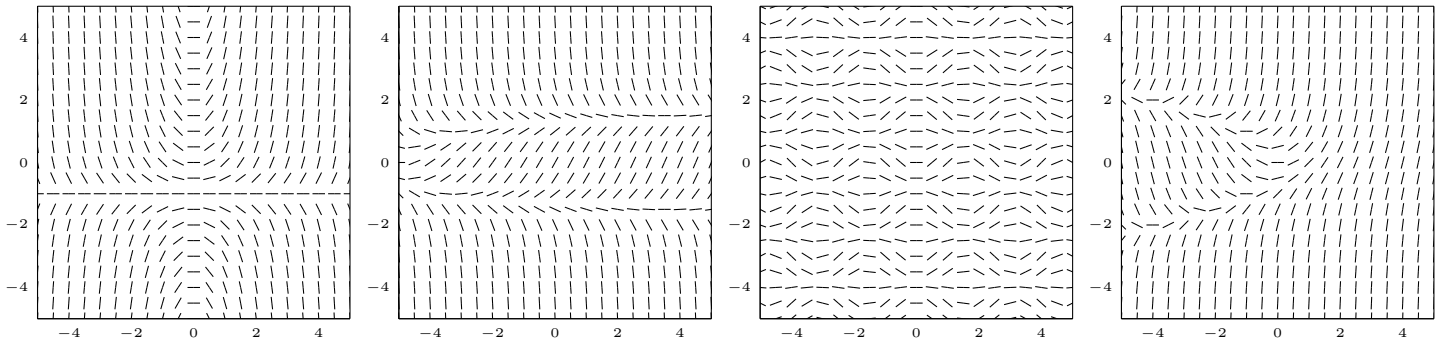
   (c) (12 pts) Solve the initial value problem by first dividing the differential equation by y^2, using the substitution $v = y^{-1}$ and then the integrating factor method. Simply plugging into formulas, not showing all relevant steps or using another method will result in zero credit. Note: this is a Bernoulli differential equation which was studied in the homework.

6. [2360/020426 (12 pts)] Write the letters a. through f. in a single column in your bluebook. Next to each letter, write the Roman numeral (or the word NONE) of the direction field corresponding to the given differential equation. No partial credit is available and no work need be shown.

   (a) $y' = y^2 + t$ (b) $y' = 2y - y^2$ (c) $y' + y^2 = \ln(t+6)$

   (d) $y' = ty + t$ (e) $y' = \sin 2y \cos 2t$ (f) $y' - \sin 2t \cos 2y = 0$

I II III IV



7. [2360/020426 (12 pts)] Consider the system of differential equations: $\begin{aligned} x' &= x^2 + y^2 - 2 \\ y' &= x^2 - y \end{aligned}$. The phase plane for this system is shown below with the origin, $(0, 0)$, at the center of the figure.

   (a) (2 pts) With regard to the system of differential equations, what does the circle represent?

   (b) (2 pts) With regard to the system of differential equations, what does the parabola represent?

   (c) (6 pts) Find all equilibrium solutions and determine their stability.

   (d) (2 pts) If the initial condition is $(x(0), y(0)) = (0, 0)$ find: i. $\lim_{t\to\infty} x(t)$ ii. $\lim_{t\to\infty} y(t)$