1. (18 points) Given the function:

$$f(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

Compute the following. If the limit does not exist, write DNE. Justification is not required for this problem.

(a) $\lim\limits_{x \to 1^-} f(x)$      (b) $\lim\limits_{x \to 1^+} f(x)$      (c) $f(1)$

(d) $\lim\limits_{x \to 2^-} f(x)$      (e) $\lim\limits_{x \to 2^+} f(x)$      (f) $\lim\limits_{x \to 2} f(x)$

**Solution:**

(a) $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x = \boxed{1}$ since $f(x) = x$ when $x < 1$.

(b) $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \left(2 - x^2\right) = 2 - 1^2 = \boxed{1}$ since $f(x) = 2 - x^2$ when $x > 1$.

(c) $f(1) = \boxed{3}$ since $f(x) = 3$ when $x = 1$.

(d) $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} \left(2 - x^2\right) = 2 - 2^2 = \boxed{-2}$ since $f(x) = 2 - x^2$ when $x \leq 2$.

(e) $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} x - 3 = 2 - 3 = \boxed{-1}$

(f) $\lim\limits_{x \to 2} f(x)$ $\boxed{\text{does not exist}}$ because $\lim\limits_{x \to 2^-} f(x) = -2$ and $\lim\limits_{x \to 2^+} f(x) = -1$ and $\lim\limits_{x \to 2^-} f(x) \neq \lim\limits_{x \to 2^+} f(x)$.

2. (24 pts) Evaluate the following limits and simplify your answers. To get full credit, be sure to show all justifications and intermediate steps. If a limit does not exist, clearly state this. If you use a theorem, clearly state its name and show that its hypothesis are satisfied. (You may not use L'Hospital's rule or dominance of powers arguments.):

(a) $\lim\limits_{x \to 2} \dfrac{x - 2}{\sqrt{4x + 1} - 3}$

(b) $\lim\limits_{t \to 0} \left( \dfrac{\sin(3t)}{5t^3 - 4t} \right)$

(c) $\lim\limits_{x \to 0} x^4 \cos \dfrac{2}{x}$

**Solution:**

(a) We start by noting that when we try to substitute $x = 2$ into $\dfrac{x - 2}{\sqrt{4x + 1} - 3}$ we get $0/0$ which means this is an indeterminant limit. To evaluate the limit we multiply the numerator and the denominator by the conjugate of $\sqrt{4x + 1} - 3$.

$$\lim_{x \to 2} \frac{x - 2}{\sqrt{4x + 1} - 3} = \lim_{x \to 2} \frac{x - 2}{\sqrt{4x + 1} - 3} \cdot \frac{\sqrt{4x + 1} + 3}{\sqrt{4x + 1} + 3}$$

$$= \lim_{x \to 2} \frac{(x - 2)\left(\sqrt{4x + 1} + 3\right)}{4x + 1 - 9}$$

$$= \lim_{x \to 2} \frac{(x - 2)\left(\sqrt{4x + 1} + 3\right)}{4x - 8}$$

$$= \lim_{x \to 2} \frac{(x - 2)\left(\sqrt{4x + 1} + 3\right)}{4(x - 2)}$$

$$= \lim_{x \to 2} \frac{\sqrt{4x + 1} + 3}{4}$$

We can now evaluate the limit by substituting $x = 2$ into $\dfrac{\sqrt{4x + 1} + 3}{4}$ and we get:

$$\lim_{x \to 2} \frac{\sqrt{4x + 1} + 3}{4} = \frac{\sqrt{4(2) + 1} + 3}{4}$$

$$= \boxed{\frac{3}{2}}$$

(b) We start by noting that when we try to substitute $t = 0$ into $\dfrac{\sin(3t)}{5t^3 - 4t}$ we get $0/0$ which means this is an indeterminant limit. To evaluate the limit we will utilize the special limit $\lim\limits_{t\to 0} \dfrac{\sin(3t)}{3t} = 1$.

$$\lim_{t\to 0} \left( \frac{\sin(3t)}{5t^3 - 4t} \right) = \lim_{t\to 0} \left( \frac{\sin(3t)}{t\left(5t^2 - 4\right)} \right)$$

$$= \lim_{t\to 0} \left( \frac{\sin(3t)}{t} \right) \cdot \left( \frac{1}{5t^2 - 4} \right)$$

$$= \lim_{t\to 0} \left( \frac{3}{3} \cdot \frac{\sin(3t)}{t} \right) \cdot \left( \frac{1}{5t^2 - 4} \right)$$

$$= \lim_{t\to 0} \left( \frac{3}{1} \cdot \frac{\sin(3t)}{3t} \right) \cdot \left( \frac{1}{5t^2 - 4} \right)$$

$$= \lim_{t\to 0} \left( \frac{3}{1} \right) \cdot \lim_{t\to 0} \left( \frac{\sin(3t)}{3t} \right) \cdot \lim_{t\to 0} \left( \frac{1}{5t^2 - 4} \right)$$

$$= 3 \cdot 1 \cdot \left( -\frac{1}{5(0)^2 - 4} \right)$$

$$= \boxed{-\frac{3}{4}}$$

(c) The limit $\lim\limits_{x\to 0} x^4 \cos \dfrac{2}{x}$ is not a $0/0$ limit and the graph of $y = \cos\left(\dfrac{2}{x}\right)$ oscillates rapidly when $x \to 0$. These two pieces of information indicate applying the squeeze theorem is appropriate. We start by recalling that:

$$-1 \le \cos\left(\frac{2}{x}\right) \le 1$$

Noting that $x^4 > 0$ for $x \neq 0$ we multiply through by $x^4$:

$$-x^4 \le x^4 \cos\left(\frac{2}{x}\right) \le x^4$$

Since $\lim\limits_{x\to 0} -x^4 = 0$ and $\lim\limits_{x\to 0} x^4 = 0$ then by the squeeze theorem $\lim\limits_{x\to 0} x^4 \cos \dfrac{2}{x} = \boxed{0}$.

3. (23 points) Let $f(x) = \cos(x) - \frac{1}{2}$

    (a) Give the domain and the range of $f(x)$ in interval notation.

    (b) Graph $f(x)$ on $[0, 2\pi]$. Be sure that you clearly label the x and y intercepts. Also, indicate where the function attains its minimum value on the interval.

    (c) Give the domain and the range of $|f(x)|$ in interval notation.

    (d) This part is unrelated to the above. Given the function $g(x) = 2\cos(x) + \sin(2x)$. Find where $g(x) \geq 0$ on the interval $[0, 2\pi]$.

**Solution:**

    (a) The domain is $\boxed{(-\infty, \infty)}$. Since the range of $\cos x$ is $[-1, 1]$, if we subtract $\frac{1}{2}$, then the range of $f$ is $\boxed{[-\frac{3}{2}, \frac{1}{2}]}$.

    (b) The graph of $f(x) = \cos(x) - \frac{1}{2}$ is given below:
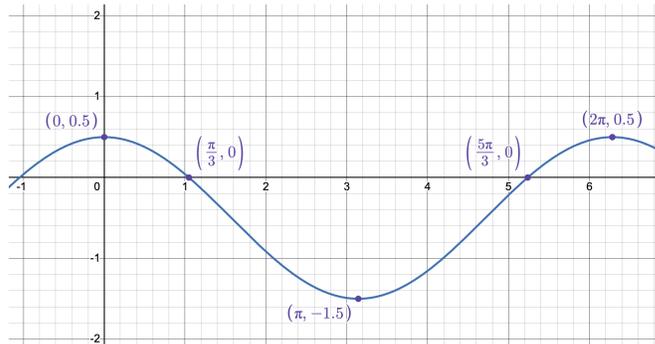


    (c) $|f(x)| = |\cos(x) - \frac{1}{2}|$. Since the range of $f(x)$ is $[-\frac{3}{2}, \frac{1}{2}]$, we find the range of $|f(x)|$ must be $\boxed{[0, \frac{3}{2}]}$.

    (d) First, we will find the zeroes of $g(x) = 2\cos(x) + \sin(2x)$.

$$2\cos(x) + \sin(2x) = 2\cos(x) + 2\sin(x)\cos(x) = 0$$
$$2\cos(x)(1 + \sin(x)) = 0$$

Let $2\cos(x) = 0$. This implies that $\cos(x) = 0$. Therefore, $x = \frac{\pi}{2}, \frac{3\pi}{2}$ are solutions.

Next, let $1 + \sin(x) = 0$. Then $\sin(x) = -1$. Therefore, $x = \frac{3\pi}{2}$. This solution is also listed above.

To solve $g(x) \leq 0$, we shall evaluate points on either side of each zero and determine the sign of the function.

$$g(0) = 2 \cdot 1 + 0 = 2 > 0$$
$$g(\pi) = 2\cos(\pi)(1 + \sin(\pi)) = -2 \cdot (1) = -2 < 0$$
$$g(2\pi) = 2\cos(2\pi)(1 + \sin(2\pi)) = 2 \cdot (1) = 2 > 0$$

Therefore, $g(x) \leq 0$ when $x$ is $\boxed{[0, \frac{\pi}{2}], [\frac{3\pi}{2}, 2\pi]}$.

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4. (20 pts) Consider the function $f(x) = \dfrac{x^2 + x}{1 + x}$ and $g(x) = \sin(2x)$.

   (a) Is $f(x)$ an even function, an odd function, or neither? Justify your answer algebraically.

   (b) Give the domain of $f(x)$ and the domain of $g(x)$ in interval notation.

   (c) Find the values where $f(x)$ is discontinuous and state which type of discontinuity it is. **Make sure to justify your answer with the appropriate limit definitions.**

   (d) Find $(f \circ g)(x)$ and state its domain within the interval $[0, 2\pi]$. Give your answer in interval notation.

   **Solution:**

   (a) To determine whether $f(x)$ is even, odd, or neither, we will evaluate $f(-x)$:

   $$f(-x) = \frac{(-x)^2 + (-x)}{1 - x} = \frac{x^2 - x}{-(x - 1)} = \frac{x(x - 1)}{-(x - 1)} = -x$$

   Since $f(-x) \neq f(x)$, which would indicate an even function, and since $f(-x) \neq -f(x)$, which would indicate an odd function, it must be that $f(x)$ is $\boxed{\text{neither}}$.

   (b) The domain of $f(x)$ is restricted by the denominator term when $x + 1 = 0$. $x = -1$ is not in the domain. Therefore, the domain of $f(x)$ is given by $\boxed{(-\infty, -1) \cup (-1, \infty)}$.

   The domain of $g(x)$ is all real numbers. In interval notation, this is $\boxed{(-\infty, \infty)}$.

   (c) By definition, $f(x)$ is continuous at $x = -1$ if, and only if, the following holds:

   $$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = f(-1)$$

   We know that $f(-1)$ is undefined because $x = -1$ is not in the domain. Therefore, we know that there must be a discontinuity at $x = -1$. We will use the limit evaluation at $x = -1$ to classify the type of discontinuity.

   $$\begin{aligned} \lim_{x \to -1} f(x) &= \lim_{x \to -1} \frac{x^2 + x}{1 + x} \\ &= \lim_{x \to -1} x \\ &= -1 \end{aligned}$$

   Since the limit at -1 exists, but the function value does not exist at that point, there is a $\boxed{\text{removable discontinuity at } x = -1}$.

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(d) Now, we find the composition of $f$ and $g$.

$$(f \circ g)(x) = f(g(x))$$
$$= f(\sin(2x))$$
$$= \frac{\sin^2(2x) + \sin(2x)}{1 + \sin(2x)}$$

Now, we consider the domain. Since division by 0 would cause this function to be undefined, we require that $1 + \sin(2x) \neq 0$. Therefore, $\sin(2x) \neq -1$.

This implies that
$$2x \neq \frac{3\pi}{2}, \frac{7\pi}{2} \implies x \neq \frac{3\pi}{4}, \frac{7\pi}{4}$$

Thus, on the interval $[0, 2\pi]$, the $\boxed{\text{domain of } (f \circ g)(x) \text{ is } [0, \frac{3\pi}{4}) \cup (\frac{3\pi}{4}, \frac{7\pi}{4}) \cup (\frac{7\pi}{4}, 2\pi]}$.

5. (15 pts) Consider the function:

$$f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x < 1 \\ a & \text{if } x = 1 \\ 2 + x & \text{if } x > 1 \end{cases}$$

Determine whether there exists a value of $a$ such that $f(x)$ is continuous at $x = 1$. Justify your answer using limits and the definition of continuity and/or determine the type of discontinuity.

**Solution:**

By definition, $f(x)$ is continuous at $x = 1$ if, and only if, the following holds:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

The limit from the left is:

$$\begin{aligned} \lim_{x \to 1^-} \frac{x^2 - 1}{x - 1} &= \lim_{x \to 1^-} \frac{(x-1)(x+1)}{(x-1)} \\ &= \lim_{x \to 1^-} x + 1 \\ &= 2 \end{aligned}$$

The limit from the right is:

$$\begin{aligned} \lim_{x \to 1^+} 2 + x &= 2 + 1 \\ &= 3 \end{aligned}$$

Since $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$, $\boxed{\text{there is a jump discontinuity at } x = 1}$.

Thus, $\boxed{\text{no value of } a \text{ exists that could make this function continuous at } x = 1}$.