1. (26 pts) Parts (a) and (b) are not related.

   (a) Let $f(x) = \frac{3}{8}x^{8/3} - \frac{18}{5}x^{5/3} + 12x^{2/3}$. Clearly state "none" if/where applicable.

   i. Identify all critical numbers of $f(x)$, if any.

   ii. Identify all intervals, if any, on which f is increasing and all intervals, if any, on which f is decreasing.

   Intervals on which f is increasing:

   Intervals on which f is decreasing:

   iii. Find the x-coordinate of every local maximum and every local minimum of f, if any. Briefly justify each answer using the First Derivative Test.

   x-coordinates of local maxima:

   x-coordinates of local minima:

Solution:

i.

$$
\begin{aligned}
f(x) &= \frac{3}{8}x^{8/3} - \frac{18}{5}x^{5/3} + 12x^{2/3} \\
f'(x) &= x^{5/3} - 6x^{2/3} + 8x^{-1/3} \\
&= x^{-1/3}(x^2 - 6x + 8) \\
&= x^{-1/3}(x-2)(x-4)
\end{aligned}
$$

$f' = 0$ at $x = 2$ and $x = 4$

f' does not exist at $x = 0$, and $x = 0$ is in the domain of f

Therefore, f has three critical numbers: $\boxed{0, 2, 4}$

ii. The following table indicates that $f' < 0$ on $(-\infty, 0) \cup (2, 4)$ and $f' > 0$ on $(0, 2) \cup (4, \infty)$.

$\sqrt[3]{x}$	$-$	$+$	$+$	$+$
$x - 2$	$-$	$-$	$+$	$+$
$x - 4$	$-$	$-$	$-$	$+$
$f'(x) = \dfrac{(x-2)(x-4)}{3\sqrt[3]{x}}$	$-$	$+$	$-$	$+$
x		$x = 0$	$x = 2$	$x = 4$

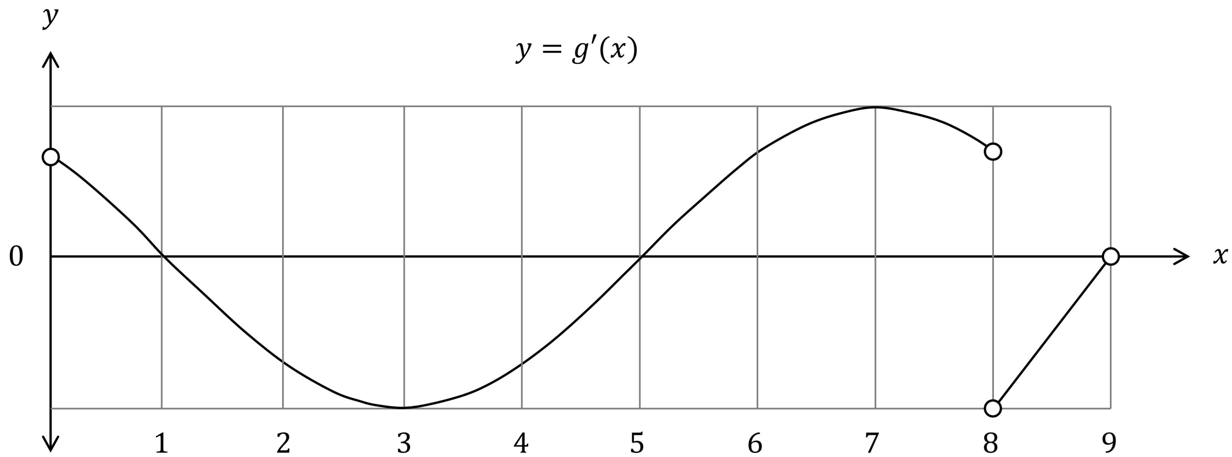
Therefore, $\boxed{f \text{ is decreasing on } (-\infty, 0) \cup (2, 4)}$ and $\boxed{f \text{ is increasing on } (0, 2) \cup (4, \infty)}$

iii. f is continuous at $x = 0$ and f transitions from decreasing to increasing at that location. Therefore, $\boxed{f \text{ has a local minimum at } x = 0}$

f is continuous at $x = 2$ and f transitions from increasing to decreasing at that location. Therefore, $\boxed{f \text{ has a local maximum at } x = 2}$

f is continuous at $x = 4$ and f transitions from decreasing to increasing at that location. Therefore, $\boxed{f \text{ has a local minimum at } x = 4}$

(b) The graph of the <u>first derivative</u> $g'(x)$ of a continuous function $g(x)$ is shown below on the interval $(0, 9)$.



Answer the following questions about the <u>continuous function</u> $g(x)$. Clearly state "none" if/where applicable.

To be clear, these questions relate to $g(x)$, which is <u>not</u> the function that is graphed above.

i. Identify all critical number(s) of $g(x)$, if any.

ii. Identify all interval(s), if any, on which $g(x)$ is increasing.

iii. Identify the x-coordinate of every local maximum of $g(x)$, if any.

iv. Identify all interval(s), if any, on which $g(x)$ is concave down.

v. Identify the x-coordinate of every inflection point of $g(x)$, if any.

Solution:

i. $g'(x) = 0$ at $x = 1$ and $x = 5$, so 1 and 5 are critical numbers of $g(x)$.

$g'(x)$ is undefined at $x = 8$ and $x = 8$ is in the domain of $g(x)$ since g is said to be continuous there. So, 8 is also a critical number of $g(x)$.

Therefore, the critical numbers of $g(x)$ on the specified interval are $x = \boxed{1, 5, 8}$

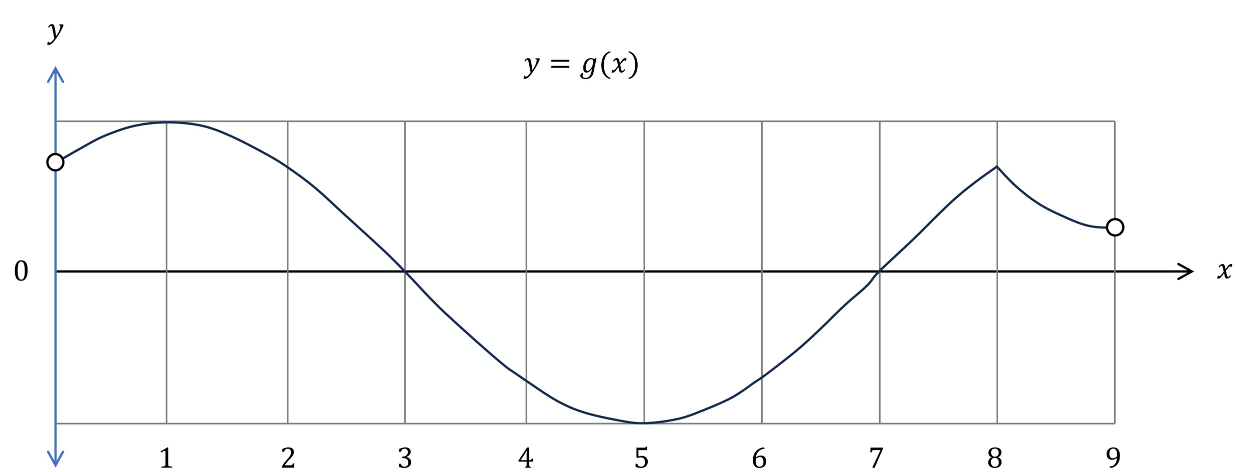
ii. $g(x)$ is increasing wherever the curve $y = g'(x)$ takes on positive values, which occurs on the intervals $\boxed{(0,1) \cup (5,8)}$

iii. According to the First Derivative Test, the continuous function $g(x)$ attains a local maximum value wherever the sign of $g'(x)$ transitions from positive to negative. That occurs at $x = \boxed{1, 8}$

iv. Concavity is determined by the second derivative of g with respect to x, which is the first derivative of g' with respect to x. So, $g(x)$ is concave down wherever $g'(x)$ is decreasing, which occurs on the intervals $\boxed{(0,3) \cup (7,8)}$

v. The continuous function $g(x)$ has an inflection point wherever its concavity changes. That occurs at values of x at which $g'(x)$ transitions between decreasing and increasing. Specifically, $g(x)$ has inflection points at $x = \boxed{3, 7, 8}$

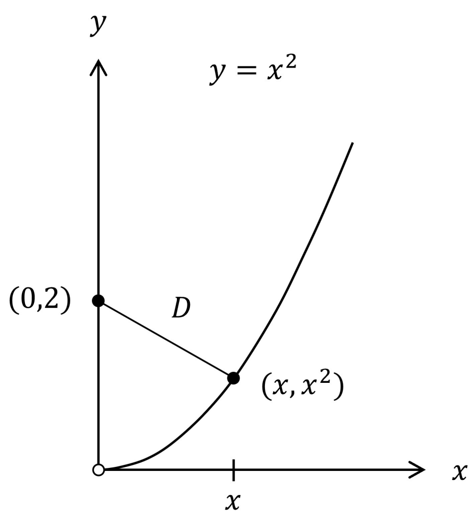
Note that $g(x)$ is said to be a continuous function. The jump discontinuity in the graph of $y = g'(x)$ at $x = 8$ indicates that $g(x)$ has a corner at $x = 8$. The graph of $y = g(x)$ behaves as follows:



2. (17 pts) Find the coordinates of the point on the curve $y = x^2$ with $x > 0$ that is closest to the point $(0, 2)$. Use the Second Derivative Test to verify that the corresponding distance is a local minimum value.

Solution:

Every positive value of x corresponds to a point (x, x^2) on the curve $y = x^2$ in Quadrant I. Let D, which is a function of x, represent the distance between the point (x, x^2) and the point $(0, 2)$, as depicted here.



The Pythagorean Theorem indicates that $D^2 = (x - 0)^2 + (x^2 - 2)^2 = x^2 + (x^2 - 2)^2$.

The minimum value of D occurs at the same value of x as the minimum value of D^2. This is because as the value of D decreases, so does the value of D^2. So, minimizing $s(x) = D^2$ is effectively equivalent to minimizing D.

$$\begin{aligned} s(x) &= D^2 = x^2 + (x^2 - 2)^2, \quad x > 0 \\ s'(x) &= 2x + 2(x^2 - 2)(2x) \\ &= 2x + 4x^3 - 8x \\ &= 4x^3 - 6x = 2x(2x^2 - 3) \end{aligned}$$

The only critical number of s on the interval $(0, \infty)$ is $x = \sqrt{3/2}$.

$s''(x) = 12x^2 - 6 = 6(2x^2 - 1)$, so that $s''(\sqrt{3/2}) = 6(2 \cdot 3/2 - 1) = 12 > 0$. Therefore, the Second Derivative Test indicates that $s(x)$, and therefore also D, attains a local minimum value at $x = \sqrt{3/2}$.

Therefore, the coordinates of the point on the curve $y = x^2$ in Quadrant I that is closest to the point $(0, 2)$ is $\boxed{\left(\sqrt{3/2}, 3/2\right)}$

Note that the First Derivative Test for Absolute Extrema could be used to formally establish that the local minimum at $x = \sqrt{3/2}$ is also the absolute minimum on the domain $(0, \infty)$.

3. (16 pts) Find an equation of the slant asymptote of the function $y = \dfrac{3x^4 - 4x^3 + x^2 + x - 1}{x^3 - 2x^2 + x - 1}$.

Solution:

The numerator of the given rational function is a polynomial of order 4 and the denominator is a polynomial of order 3. Since the order of the numerator is exactly one greater than the order of the denominator, the rational function has a slant asymptote. To find the equation of that asymptote, perform polynomial long division.

$$
\begin{array}{r|r}
 & 3x + 2 \\
\hline
x^3 - 2x^2 + x - 1 & 3x^4 - 4x^3 + \ \ x^2 + \ \ x - 1 \\
 & 3x^4 - 6x^3 + 3x^2 - 3x \\
\cline{2-2}
 & 2x^3 - 2x^2 + 4x - 1 \\
 & 2x^3 - 4x^2 + 2x - 2 \\
\cline{2-2}
 & 2x^2 + 2x + 1
\end{array}
$$

So, the given improper rational function can be expressed as the following sum of a polynomial and a proper rational function.

$$\frac{3x^4 - 4x^3 + x^2 + x - 1}{x^3 - 2x^2 + x - 1} = 3x + 2 + \frac{2x^2 + 2x + 1}{x^3 - 2x^2 + x - 1}$$

Therefore, the equation of the slant asymptote, which is the quotient that was obtained by polynomial long division, is $\boxed{y = 3x + 2}$

4. (20 pts) Suppose the first and second derivatives of a continuous function $h(x)$ on the interval $[0, 5]$ are as follows:

$$h'(x) = (x-1)(x-4)^3$$
$$h''(x) = (x-4)^2(4x-7)$$

For each of the following questions, clearly state "none" if/where applicable.

(a) Identify all critical numbers of h on the interval $[0, 5]$.

Solution:

$$h'(x) = (x-1)(x-4)^3 = 0$$
$$x = \boxed{1, 4}$$

(b) Identify all intervals, if any, on which h is increasing and all intervals, if any, on which h is decreasing on the interval $[0, 5]$.

Subintervals of $[0, 5]$ on which h is increasing:

Subintervals of $[0, 5]$ on which h is decreasing:

Solution:

The following table indicates that $h' < 0$ on $(1, 4)$ and $h' > 0$ on $[0, 1) \cup (4, 5]$.

	$x=0$ to $x=1$	$x=1$ to $x=4$	$x=4$ to $x=5$
$(x-1)$	$-$	$+$	$+$
$(x-4)^3$	$-$	$-$	$+$
$h'(x) = (x-1)(x-4)^3$	$+$	$-$	$+$

$x = 0$ $x = 1$ $x = 4$ $x = 5$

Therefore, $\boxed{h \text{ is decreasing on } (1, 4)}$ and $\boxed{h \text{ is increasing on } [0, 1) \cup (4, 5]}$

(c) Find the x-coordinate of every local maximum and every local minimum of h, if any, on the interval $[0, 5]$. Briefly justify each answer using the First Derivative Test.

x-coordinates of local maxima on the interval $[0, 5]$:

x-coordinates of local minima on the interval $[0, 5]$:

Solution:

From part (b) we know that the continuous function h transitions from increasing to decreasing at $x = 1$ and that h transitions from decreasing to increasing at $x = 4$. Therefore, the First Derivative Test indicates that h attains a $\boxed{\text{local maximum at } x = 1}$ and h attains a $\boxed{\text{local minimum at } x = 4}$

(d) Identify all intervals, if any, on which h is concave up and all intervals, if any, on which h is concave down on the interval $[0, 5]$.

i. Subintervals of $[0, 5]$ on which h is concave up:

ii. Subintervals of $[0, 5]$ on which h is concave down:

Solution:

Recall that $h''(x) = (x - 4)^2(4x - 7)$. Since $(x - 4)^2$ can not be negative, the sign of h'' is determined solely by the sign of $(4x - 7)$.

Since $(4x - 7) < 0$ for $x < 7/4$, h is $\boxed{\text{concave down on the interval } (0, 7/4)}$

Since $(4x - 7) > 0$ for $x > 7/4$, h is $\boxed{\text{concave up on the interval } (7/4, 5)}$

Note that $h''(4) = 0$. From part (c), we know that h is differentiable and attains a local minimum at $x = 4$. Therefore, the curve of h lies above its tangent lines at $x = 4$ so that it is concave up there.

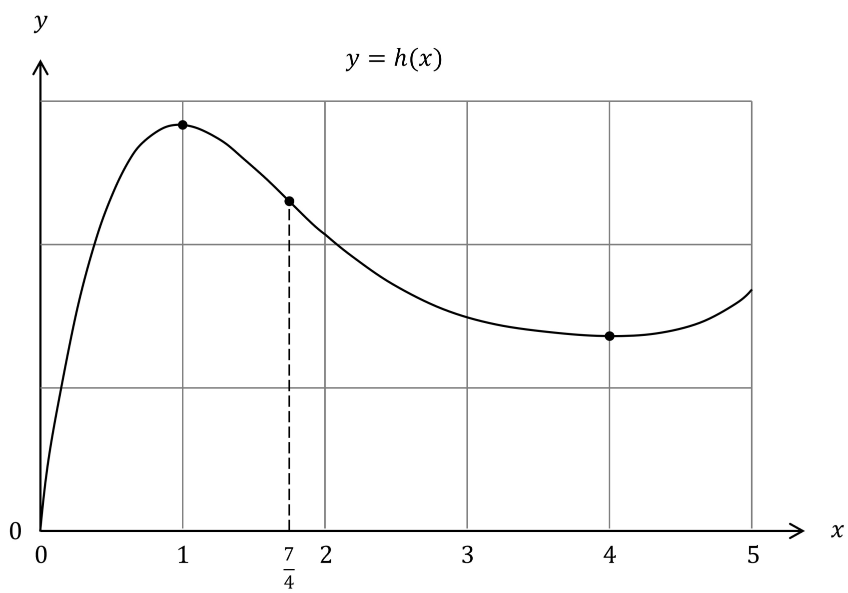
(e) Find the x-coordinate of every inflection point of h on $[0, 5]$, if any.

Solution:

Since h is continuous on $[0, 5]$ h has an inflection point at every x value on that interval at which the concavity changes. The concavity changes only at $x = 7/4$ so that the only inflection point of f is located at $\boxed{x = 7/4}$

(f) Sketch $y = h(x)$ on $[0, 5]$ on the axis system below, assuming that $h(0) = 0$ and $h(x) > 0$ on $(0, 5]$. Your sketch should clearly indicate the x-coordinate of each local maximum, local minimum, and inflection point. The y-coordinates of those points are unimportant, although the behavior of the function curve should be clearly consistent with your answers to parts (a) - (e).

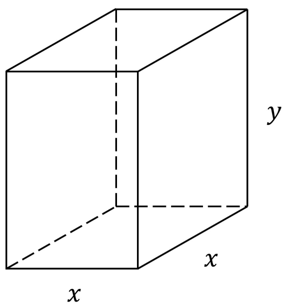
Solution:



5. (21 pts) A rectangular box with a square base and a closed top is to have a volume of 30 m^3. Material for the base and the top costs \$3 per square meter and material for the sides costs \$2 per square meter. Find the length of the base of the box that minimizes the total material cost of the box, including the correct unit of measurement. Use the Second Derivative Test to verify that the result is a local minimum value.

Solution:

Below is a sketch of the box. Since it has a square base, the variable x has been assigned to represent both the length and width of the box. The variable y has been assigned to represent the box's height.



The surface area of both the top and the base equals x^2, for a combined surface area of $2x^2$ square meters.

The surface area of each of the four sides equals xy, for a combined surface area of $4xy$ square meters.

The combined cost of the top and the base equals $(2x^2 \text{ sq meters})(\$3/\text{sq meter}) = 6x^2$ dollars.

The combined cost of the four sides equals $(4xy \text{ sq meters})(\$2/\text{sq meter}) = 8xy$ dollars.

Therefore, the total material cost of the box is $(6x^2 + 8xy)$ dollars.

The volume of the box must equal 30 m^3 so that $x^2y = 30$. So, $y = 30x^{-2}$, which can be used to construct the following objective function for the total cost, solely in terms of x:

$$\begin{aligned} C(x) &= 6x^2 + 8xy \\ &= 6x^2 + (8x)(30x^{-2}) \\ &= 6x^2 + 240x^{-1}, \quad x > 0 \end{aligned}$$

$$\begin{aligned} C'(x) &= 12x - 240x^{-2} \\ &= 12x^{-2}(x^3 - 20) \end{aligned}$$

The only critical number of $C(x)$ on its domain is $\boxed{x = \sqrt[3]{20} \text{ meters}}$

$C''(x) = 12 + 480x^{-3}$ which is positive for all $x > 0$. Therefore, the Second Derivative Test confirms that the preceding critical number is a local minimum.