Department of Applied Mathematics
MATHEMATICAL STATISTICS PRELIMINARY EXAMINATION
January 2026

Instructions:

- Do four of the five problems.
- Place an **X** on the line next to the problem number you are **NOT** submitting for grading.
- Do not write your name anywhere on this exam.
- Write your student number on each page submitted for grading.
- Show all relevant work – correct answers without adequate justification will receive no credit!

1. _________ 2. _________ 3. _________ 4. _________ 5. _________ Total _________

Student Number __________________________

Problem 1. (25 points)

Let $X_1, \ldots, X_n$ be iid $f(x;\theta) = (2\theta)^{-1} e^{-|x|/\theta}$ for all $x \in \mathbb{R}$.

(a) Show $\mathbb{E}|X| = \theta$ and $\mathbb{E}(X^2) = 2\theta^2$.

(b) Show $\frac{1}{n}\sum_{i=1}^n |X_i|$ is unbiased for θ.

(c) Find the Fisher information for θ based on one sample, and based on n samples.

(d) Determine whether your estimator from (b) attains the Cramér-Rao lower bound.

Problem 2. (25 points)

Suppose Z_1, Z_2 and Z_3 are iid $N(0,1)$ random variables. Define

$$X_1 = 2Z_1 + Z_2 - 3, \quad X_2 = aZ_1 + bZ_2 + cZ_3, \quad X_3 = Z_1 - Z_2 + Z_3 + 3$$

for some constants a, b, c.

(a) Explicitly write the joint pdf for (X_1, X_2, X_3).

(b) Determine conditions on a, b, c such that X_1 and X_2 are independent.

(c) Find the distribution of the random vector (Y_1, Y_2) where

$$Y_1 = X_1 - X_2, \quad Y_2 = X_1 + X_3.$$

Problem 3. (25 points)

(a) Carefully define what it means for a sequence $\{X_n\}_{n=1}^{\infty}$ to be bounded in probability.

(b) Prove that $X_n = O_P(1)$ if there is a $p > 0$ such that $\mathbb{E}(|X_n|^p)$ is bounded for $n \geq 1$.

Problem 4. (25 points)

Let $X_1, \ldots, X_n$ be iid $N(\theta, 1)$ random variables.

(a) Show that the joint distribution of $X_1, \ldots, X_n$ is a member of the exponential family.

(b) Find the unique UMVUE for θ.

(c) Find the joint distribution of $\overline{X}$ and $X_1 - \overline{X}$.

(d) Find the unique UMVUE, $\hat{\tau}_n(\theta)$, for $\tau(\theta) = P(X_1 < c)$ where c is a constant.
(Hint: recall that $\mathbb{1}_{[X_1 < c]}$ is unbiased for $\tau(\theta)$).

(e) Write $\hat{\tau}_n(\theta)$ in terms of Φ, the cdf of a $N(0, 1)$.

Problem 5. (25 points)

Suppose $X_1, X_2, \ldots, X_n$ are iid $Exp(rate = \lambda)$.

(a) Find the distribution of $X_{(1)}$, the sample minimum.

(b) Based on $X_{(1)}$, find the exact 95% confidence interval for λ, whose lower bound is zero.

(c) Find the UMP test for $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda > \lambda_0$ at significance level α. Explicitly write the rejection region in terms of a known cdf or quantile function.
