Preliminary Exam
Partial Differential Equations
9AM - 12PM, Wed Jan 7, 2026

Student ID (do NOT write your name):

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#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

There are five problems. **Solve four of the five problems.**
Each problem is worth 25 points.
A sheet of convenient formulae is provided.

1. **Method of Characteristics.** Solve

$$u_t + 2u_x = 1, \qquad t > 0,$$

with data prescribed on the finite oblique segment

$$\Gamma : \quad (x,t) = (s,\ 1-s), \qquad 0 \le s \le 1,$$

by

$$u(s, 1-s) = \sin(\pi s).$$

(a) (8 points) Check the transversality condition for Γ.
Solution: The characteristic equations are

$$\frac{dt}{d\tau} = 1, \qquad \frac{dx}{d\tau} = 2,$$

and the parameterized initial data are

$$(x_0(s), t_0(s)) = (s,\ 1-s), \qquad 0 \le s \le 1, \qquad u(x_0(s), t_0(s)) = \sin(\pi s).$$

The characteristic map is

$$(x,t) = (x(s,\tau), t(s,\tau)) = \big(x_0(s) + 2(\tau - t_0(s)),\ \tau\big),$$

so

$$\frac{\partial(x,t)}{\partial(s,\tau)} = \begin{pmatrix} \partial_s x & \partial_\tau x \\ \partial_s t & \partial_\tau t \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \qquad J = \det \frac{\partial(x,t)}{\partial(s,\tau)} = 3 \ne 0.$$

Hence Γ satisfies the transversality condition for all $0 \le s \le 1$.

(b) (10 points) Find the characteristic curves and compute u along them.
Solution: Taking $\tau = t$, we find the characteristics satisfy

$$x - 2t = C,$$

and along each characteristic

$$\frac{du}{d\tau} = 1 \quad \Rightarrow \quad u = t + C_1.$$

To connect to the data on Γ, label characteristics by the intersection point

$$(x_0, t_0) = (s,\ 1-s), \qquad 0 \le s \le 1.$$

The characteristic through $(s, 1-s)$ has constant

$$C = x_0 - 2t_0 = s - 2(1-s) = 3s - 2,$$

so it is

$$x - 2t = 3s - 2.$$

On this curve, using $u(s, 1-s) = \sin(\pi s)$ and $u(t) = u(t_0) + (t - t_0)$,

$$u(x,t) = \sin(\pi s) + \big(t - (1-s)\big) = t - 1 + s + \sin(\pi s),$$

for points (x, t) lying on the characteristic $x - 2t = 3s - 2$. We thus find

$$u(x,t) = \frac{x+t-1}{3} + \sin\left[\frac{x-2t+2}{3}\right].$$

(c) (7 points) Determine explicitly the region in the (x,t)-plane where the solution is uniquely defined, and describe what fails outside this region.

Solution: Let (x, t) be a point whose backward characteristic meets Γ at $(x_0, t_0) = (s, 1-s)$. Since $x - 2t$ is constant along characteristics,

$$x - 2t = x_0 - 2t_0 = s - 2(1-s) = 3s - 2,$$

so

$$s = \frac{x-2t+2}{3}.$$

Thus the backward characteristic hits Γ if and only if $s \in [0, 1]$, i.e.

$$0 \le \frac{x-2t+2}{3} \le 1 \quad \Longleftrightarrow \quad -2 \le x - 2t \le 1.$$

Moreover, we must be *forward* of the data point, i.e. $t \ge t_0 = 1 - s$, which becomes

$$t \ge 1 - \frac{x-2t+2}{3} \quad \Longleftrightarrow \quad t \ge 1 - x.$$

Therefore the solution is uniquely determined on the region

$$\mathcal{R} = \{(x,t) :\ t \ge 0,\ -2 \le x - 2t \le 1,\ t \ge 1 - x\}.$$

Outside $\mathcal{R}$, the characteristic through (x, t) does not intersect the initial data Γ, so the given data do not determine $u(x, t)$ uniquely.

2. **Heat Equation.** Let u be a classical solution on the closure of $\Omega = (0,1) \times (0, \infty)$ to the problem

$$\begin{cases} u_t = u_{xx}, & (x,t) \in \Omega, \\ u(x,0) = x(1-x), & 0 \le x \le 1, \\ u(0,t) = u(1,t) = 0 & t > 0. \end{cases}$$

(a) (7 points) Prove that u is non-negative.
Solution: Let $\Gamma_T = (0,1) \times \{t = 0\} \cup \{x = 0, 1\} \times [0, T)$. By the minimum principle, we know that

$$\min_{(x,t)\in[0,1]\times[0,T]} u(x,t) \geq \min_{(x,t)\in\Gamma_T} u(x,t) \geq 0$$

for any $T > 0$, and so it holds in Ω.

(b) (12 points) Determine the maximum range of positive a, b so that

$$u(x,t) \leq w(x,t) = ax(1-x)e^{-bt}.$$

Solution: Let $v = w - u$. Our goal is to find a, b such that we can apply the maximum principle and get that $v \geq 0$ in Ω. Note that if $v_t \geq v_{xx}$, then we can apply the maximum principle. Now, we have that

$$w(x,0) = ax(1-x), \ \ w_t(x,t) = -abx(1-x)e^{-bt}\text{ , and } w_{xx} = -2ae^{-bt}.$$

Moreover, thanks to linearity, we have

$$v_t - v_{xx} = (w_t - w_{xx}) - (u_t - u_{xx}) = a[2 - bx(1-x)]e^{-bt} \geq a\left(2 - \frac{b}{4}\right)e^{-bt}, \tag{1}$$

since the max of $x(1-x)$ occurs at $x = 1/2$. Thus, we only need to figure out when $a\left(2 - \frac{b}{4}\right) \geq 0$. This implies that we need $a > 0$ and $0 < b \leq 8$. Now, we need to look at the boundary of the parabolic domain. Note that

$$v(0,t) = v(1,t) = 0$$

and $v(x,0) = (a-1)x(1-x)$, which is nonnegative if $a \geq 1$. In summary, we need that $0 < b \leq 8$ and $a \geq 1$.

(c) (6 points) Show that $u \to 0$ uniformly in x as $t \to \infty$.
Solution: Fix $a = 1$ and $b = 8$ we see that

$$u(x,t) \leq \frac{1}{4}e^{-8t}$$

for all $(x,t) \in \Omega$. For fixed x, taking the limit as $t \to \infty$ we see that $u(x,t) \to 0$ at a rate independent of x. Thus, the convergence is uniform.

3. **Separation of variables.**

Consider the BVP

$$\begin{cases} \Delta u = f(x), & x \in \Omega, \\ \frac{\partial u}{\partial n} = g(x), & x \in \partial\Omega, \end{cases}$$

where f, g are smooth functions.

(a) (5 points) Find the compatability conditions on f and g for the possibility of a solution.
Solution: Integrate the equation to obtain that

$$\int_\Omega f(x)\,dx = \int_\Omega \Delta u(x)\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial n}\,dS_x = \int_{\partial\Omega} g(x)\,dS_x.$$

(b) (15 points) Solve the specific problem

$$\begin{cases} u_t = u_{xx} + \cos(x), & x \in (0, 2\pi), \\ u_x(0,t) = u_x(2\pi, t) = 0 & t > 0, \\ u(x,0) = \cos(x) + \cos(2x), & x \in [0, 2\pi]. \end{cases}$$

Solution: First we perform a change of variables, $z = u - \cos(x)$, and so z satisfied

$$\begin{cases} z_t = z_{xx}, & x \in (0, 2\pi), \\ z_x(0,t) = z_x(2\pi, t) = 0 & t > 0, \\ z(x,0) = \cos(2x), & x \in [0, 2\pi]. \end{cases}$$

Now, let $z(x,t) = V(x)W(t)$ and plug into the pde for z to obtain that

$$\frac{W'(t)}{W(t)} = \frac{V''(x)}{V(x)} = -\lambda.$$

For $\lambda \geq 0$ we get that

$$V(x) = a_n \cos(\sqrt{\lambda}x) + b_n \sin(\sqrt{\lambda}x).$$

Taking the derivative, we have that

$$V'(x) = -a_n\sqrt{\lambda}\sin(\sqrt{\lambda}x) + b_n\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

Using the boundary conditons, we see that $V'(0) = 0$ implies that $b_n = 0$. Moreover, $V'(2\pi) = a_n\sqrt{\lambda}\sin(2\pi\sqrt{\lambda}) = 0$ give the eigenvalues $\lambda_n = \left(\frac{n}{2}\right)^2$ and eigenfunctions $V_n(x) = \cos(\sqrt{\lambda_n}x)$ for $n \geq 0$. We can now solve for T for given λ_n to get

$$T_n(t) = c_n e^{-\left(\frac{n}{2}\right)^2 t}.$$

Putting this all together, we have that

$$z(x,t) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{nx}{2}\right) e^{-\left(\frac{n}{2}\right)^2 t}.$$

To find c_n we use the initial condition,

$$z(x,0) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{nx}{2}\right) = \cos(2x),$$

from which we can see that $c_4 = 1$ and $c_n = 0$ for $n \neq 4$. Then, $z(x,t) = \cos(2x)\, e^{-4t}$ and switching back to u, you obtain that

$$u(x,t) = \cos(2x)\, e^{-4t} + \cos(x).$$

(c) (5 points) Does there exist a solution to the equilibrium version of the problem in (b) if we change its boundary conditions to

$$u_x(0,t) = 1 \quad \text{and} \quad u_x(2\pi, t) = 0$$

Solution: We use the condition that we found in part (a):

$$0 = \int_0^{2\pi} \cos(x)\, dx \neq -1 = u_x|_0^{2\pi}.$$

The compatibility condition is not satisfied, so there cannot be a solution.

4. **Laplace's equation.**

(a) (4 points) State the mean-value property for subharmonic v.
Solution:

Theorem 1 (Mean-value property for subharmonic functions)**.** Let $U \subset \mathbb{R}^n$ be open and let $v \in C^2(U)$ be subharmonic, i.e. $\Delta v \geq 0$ in U. Fix $x_0 \in U$ and $r > 0$ such that $\overline{B_r(x_0)} \subset U$. Then

$$v(x_0) \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(x_0)} v \, dS, \tag{2}$$

$$v(x_0) \leq \frac{1}{|B_r|} \int_{B_r(x_0)} v \, dx. \tag{3}$$

Here $|B_r|$ is the volume of $B_r(x_0)$ and $|\partial B_r|$ is the surface area of $\partial B_r(x_0)$.

(b) (13 points) Assume that for every $x_0 \in U$ and every $r > 0$ such that $\overline{B_r(x_0)} \subset U$,

$$v(x_0) = \frac{1}{|\partial B_r|} \int_{\partial B_r(x_0)} v \, dS.$$

Show that v is harmonic in U, i.e. $\Delta v = 0$. *Hint:* For $v \in C^2(U)$ and $x_0 \in U$, define the spherical mean

$$M(r) := \frac{1}{|\partial B_r|} \int_{\partial B_r(x_0)} v \, dS.$$

You may want to use the identity

$$M'(r) = \frac{1}{|\partial B_r|} \int_{B_r(x_0)} \Delta v \, dx.$$

Solution: Fix $x_0 \in U$. Choose $r_0 > 0$ so that $\overline{B_{r_0}(x_0)} \subset U$. For $0 < r < r_0$ define the spherical mean $M(r)$ as in the hint. By hypothesis, $M(r) = v(x_0)$ for all $0 < r < r_0$, hence $M'(r) = 0$. On the other hand, the differentiation identity for spherical means that was provided in the hint states that for $u \in C^2$,

$$M'(r) = \frac{1}{|\partial B_r|} \int_{B_r(x_0)} \Delta v \, dx,$$

implies that

$$\int_{B_r(x_0)} \Delta v \, dx = 0 \qquad \text{for all } 0 < r < r_0.$$

Let $f := \Delta v$, which is continuous on U. Divide by $|B_r|$ to obtain

$$0 = \frac{1}{|B_r|} \int_{B_r(x_0)} f(x) \, dx.$$

Letting $r \to 0^+$ and using continuity of f (the average of a continuous function over shrinking balls tends to its value at the center) yields

$$0 = \lim_{r \to 0^+} \frac{1}{|B_r|} \int_{B_r(x_0)} f(x) \, dx = f(x_0) = \Delta v(x_0).$$

Since $x_0 \in U$ was arbitrary, $\Delta v = 0$ in U, so v is harmonic.

(c) (4 points) Let $u \in C^2(U)$ be a harmonic function in $U \subset \mathbb{R}^n$ is open. Assume $F \in C^2(\mathbb{R})$ is convex and let $w = F(u)$. Prove that w is subharmonic.
Solution: $\Delta w = F''(u)|\nabla u|^2 + F'(u)\Delta u = F''(u)|\nabla u|^2 \geq 0$.

(d) (4 points) Assume $U = \mathbb{R}^n$ and $u \in L^2(U)$ then prove $u \equiv 0$.
Solution: Take $x_0 \in U$ and $r > 0$. Let $F(s) = s^2$, a convex function, so by part (c) u^2 is subharmonic because u is assumed to be harmonic. Now, by part (a), we see that

$$u^2(x_0) \leq \frac{1}{|B_r|}\int_{B_r(x_0)} u^2\,dx \leq \frac{||u||_{L^2}^2}{|B_r|} \to 0$$

as $r \to \infty$. Since x_0 as arbitrary, we conclude that $u \equiv 0$.

5. **Wave equation.** Let u be a classical solution of

$$u_{tt} = u_{xx}, \qquad 0 < x < L,\ t > 0,$$

with initial data $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, and boundary conditions

$$u_x(L,t) + \delta u_t(L,t) = 0, \qquad u_x(0,t) - \alpha u_t(0,t) = \beta\, u_t(L,t), \qquad t > 0,$$

where $\alpha > 0$, $\delta > 0$, and $\beta \in \mathbb{R}$ are constants.

(a) (10 points) Define the energy

$$E(t) = \frac{1}{2}\int_0^L \left(u_t(x,t)^2 + u_x(x,t)^2\right) dx.$$

Show that

$$E'(t) = -\delta\, u_t(L,t)^2 - \alpha\, u_t(0,t)^2 - \beta\, u_t(0,t)u_t(L,t).$$

Solution: Differentiate:

$$E'(t) = \int_0^L \left(u_t u_{tt} + u_x u_{xt}\right) dx.$$

Using $u_{tt} = u_{xx}$ and integrating by parts in x,

$$\int_0^L u_t u_{tt}\,dx = \int_0^L u_t u_{xx}\,dx = \Big[u_t u_x\Big]_0^L - \int_0^L u_{xt}u_x\,dx.$$

Hence the bulk terms cancel with $\int_0^L u_x u_{xt}\,dx$, yielding

$$E'(t) = \Big[u_t u_x\Big]_0^L = u_t(L,t)u_x(L,t) - u_t(0,t)u_x(0,t).$$

Using the boundary conditions

$$u_x(L,t) = -\delta u_t(L,t), \qquad u_x(0,t) = \alpha u_t(0,t) + \beta u_t(L,t),$$

we obtain

$$\begin{aligned} E'(t) &= u_t(L,t)\big(-\delta u_t(L,t)\big) - u_t(0,t)\big(\alpha u_t(0,t) + \beta u_t(L,t)\big) \\ &= -\delta\, u_t(L,t)^2 - \alpha\, u_t(0,t)^2 - \beta\, u_t(0,t)u_t(L,t). \end{aligned}$$

(b) (8 points) Use (a) to prove uniqueness of classical solutions in the case $\beta = 0$ (with $\delta \geq 0$ fixed).

Solution: Let u and v be two classical solutions with the same data and set $w = u - v$. Then $w_{tt} = w_{xx}$, $w(\cdot, 0) = 0$, $w_t(\cdot, 0) = 0$, and when $\beta = 0$ the boundary conditions reduce to

$$w_x(L, t) + \delta w_t(L, t) = 0, \qquad w_x(0, t) - \alpha w_t(0, t) = 0.$$

Applying the energy identity from part (a) to w yields

$$E_w'(t) = -\delta\, w_t(L, t)^2 - \alpha\, w_t(0, t)^2 \leq 0.$$

Since $E_w(t) \geq 0$ and $E_w(0) = 0$, it follows that $E_w(t) \equiv 0$ for all $t \geq 0$. Hence $w_t \equiv 0$ and $w_x \equiv 0$ on $(0, L) \times (0, \infty)$, so w is constant in both x and t. Using $w(\cdot, 0) = 0$, we conclude that $w \equiv 0$, and therefore $u \equiv v$.

(c) (7 points) Assume $\beta \neq 0$ (and $\delta > 0$). Using the formula for $E'(t)$ from part (a), show that if

$$\beta^2 \leq 4\alpha\delta,$$

then the energy is nonincreasing:

$$E'(t) \leq 0 \qquad \text{for all } t \geq 0.$$

Solution: From part (a),

$$E'(t) = -\delta\, u_t(L, t)^2 - \alpha\, u_t(0, t)^2 - \beta\, u_t(0, t) u_t(L, t).$$

Let $a = u_t(0, t)$ and $b = u_t(L, t)$. Then

$$E'(t) = -\alpha a^2 - \delta b^2 - \beta ab.$$

Complete the square:

$$-\alpha a^2 - \delta b^2 - \beta ab = -\alpha \left(a + \frac{\beta}{2\alpha} b \right)^2 - \left(\delta - \frac{\beta^2}{4\alpha} \right) b^2.$$

If $\beta^2 \leq 4\alpha\delta$, then $\delta - \beta^2/(4\alpha) \geq 0$, so both terms on the right-hand side are ≤ 0. Hence $E'(t) \leq 0$ for all $t \geq 0$.