Preliminary Exam
Partial Differential Equations
9AM - 12PM, Wed Jan 7, 2026

Student ID (do NOT write your name):

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#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

There are five problems. **Solve four of the five problems.**
Each problem is worth 25 points.
A sheet of convenient formulae is provided.

1. **Method of Characteristics.** Solve

$$u_t + 2u_x = 1, \qquad t > 0,$$

with data prescribed on the finite oblique segment

$$\Gamma : \quad (x, t) = (s,\ 1 - s), \qquad 0 \le s \le 1,$$

by

$$u(s, 1 - s) = \sin(\pi s).$$

(a) (8 points) Check the transversality condition for Γ.

(b) (10 points) Find the characteristic curves and compute u along them.

(c) (7 points) Determine explicitly the region in the (x, t)-plane where the solution is uniquely defined, and describe what fails outside this region.

2. **Heat Equation.** Let u be a classical solution on the closure of $\Omega = (0, 1) \times (0, \infty)$ to the problem

$$\begin{cases} u_t = u_{xx}, & (x,t) \in \Omega, \\ u(x, 0) = x(1 - x), & 0 \le x \le 1, \\ u(0, t) = u(1, t) = 0 & t > 0. \end{cases}$$

(a) (7 points) Prove that u is non-negative.

(b) (12 points) Determine the maximum range of positive a, b so that

$$u(x, t) \le w(x, t) = ax(1 - x)e^{-bt}.$$

(c) (6 points) Show that $u \to 0$ uniformly in x as $t \to \infty$.

3. **Separation of variables.**

Consider the BVP

$$\begin{cases} \Delta u = f(x), & x \in \Omega, \\ \frac{\partial u}{\partial n} = g(x), & x \in \partial\Omega, \end{cases}$$

where f, g are smooth functions.

(a) (5 points) Find the compatability conditions on f and g for the possibility of a solution.

(b) (15 points) Solve the specific problem

$$\begin{cases} u_t = u_{xx} + \cos(x), & x \in (0, 2\pi), \\ u_x(0,t) = u_x(2\pi, t) = 0 & t > 0, \\ u(x,0) = \cos(x) + \cos(2x), & x \in [0, 2\pi]. \end{cases}$$

(c) (5 points) Does there exist a solution to the equilibrium version of the problem in (b) if we change its boundary conditions to

$$u_x(0,t) = 1 \quad \text{and} \quad u_x(2\pi, t) = 0$$

4. **Laplace's equation.**

(a) (4 points) State the mean-value property for subharmonic v.

(b) (13 points) Assume that for every $x_0 \in U$ and every $r > 0$ such that $\overline{B_r(x_0)} \subset U$,

$$v(x_0) = \frac{1}{|\partial B_r|} \int_{\partial B_r(x_0)} v \, dS.$$

Show that v is harmonic in U, i.e. $\Delta v = 0$. *Hint:* For $v \in C^2(U)$ and $x_0 \in U$, define the spherical mean

$$M(r) := \frac{1}{|\partial B_r|} \int_{\partial B_r(x_0)} v \, dS.$$

You may want to use the identity

$$M'(r) = \frac{1}{|\partial B_r|} \int_{B_r(x_0)} \Delta v \, dx.$$

(c) (4 points) Let $u \in C^2(U)$ be a harmonic function in $U \subset \mathbb{R}^n$ is open. Assume $F \in C^2(\mathbb{R})$ is convex and let $w = F(u)$. Prove that w is subharmonic.

(d) (4 points) Assume $U = \mathbb{R}^n$ and $u \in L^2(U)$ then prove $u \equiv 0$.

5. **Wave equation.** Let u be a classical solution of

$$u_{tt} = u_{xx}, \qquad 0 < x < L, \ t > 0,$$

with initial data $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, and boundary conditions

$$u_x(L,t) + \delta u_t(L,t) = 0, \qquad u_x(0,t) - \alpha u_t(0,t) = \beta \, u_t(L,t), \qquad t > 0,$$

where $\alpha > 0$, $\delta > 0$, and $\beta \in \mathbb{R}$ are constants.

(a) (10 points) Define the energy

$$E(t) = \frac{1}{2} \int_0^L \left(u_t(x,t)^2 + u_x(x,t)^2 \right) dx.$$

Show that

$$E'(t) = -\delta \, u_t(L,t)^2 - \alpha \, u_t(0,t)^2 - \beta \, u_t(0,t) u_t(L,t).$$

(b) (8 points) Use (a) to prove uniqueness of classical solutions in the case $\beta = 0$ (with $\delta \geq 0$ fixed).

(c) (7 points) Assume $\beta \neq 0$ (and $\delta > 0$). Using the formula for $E'(t)$ from part (a), show that if

$$\beta^2 \leq 4\alpha\delta,$$

then the energy is nonincreasing:

$$E'(t) \leq 0 \qquad \text{for all } t \geq 0.$$