Write your name and your professor's name or your section number below. You are *not* allowed to use textbooks, notes, or a calculator. To receive full credit on a problem you must show **sufficient justification for your conclusion** unless explicitly stated otherwise.

Name:

Instructor and Section:

1. (24 pts) If the statement is **always true**, write "TRUE"; if it is possible for the statement to be false then mark "FALSE." You must give a **justification** for your answer. That is, if the answer is true, provide a brief proof. If the answer is false, provide a counterexample.

   (a) If you reorder the basis vectors before starting the Gram-Schmidt process you will get the same basis.

   (b) If A is non-singular, then both A^TA and AA^T are positive definite.

   (c) The product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)x dx$ is a valid inner product on C^0.

   (d) The image of a square matrix is orthogonal to its kernel when using the dot product.

Solution

(a) False. A counterexample is

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \text{ vs } \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

The first set produces

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

while the second produces

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

(b) True. If A is non-singular, then it is square and full rank, so the columns are linearly independent and A^TA is positive definite. Since the rank of A^T equals the rank of A, the same is true for A^T and hence $AA^T = (A^T)^TA^T$ is positive definite.

(c) False. It fails positivity. For example setting $f(x) = g(x) = 1$, gives $\langle 1, 1\rangle = 0$.

(d) False. A counterexample is

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

which has a kernel with basis $(1, -1)^T$, but the image basis $(1, 2)^T$.

2. (20 pts) The following questions are unrelated.

   (a) (5 pts) Write out explicitly the Cauchy-Schwartz inequality for the the L^2 inner product on the interval $[0, 2]$. Show that it is valid for the functions $f(x) = x$ and $g(x) = x^2$.

   (b) (5 pts) Compute the L^∞ norms of the functions f and g from (a) on the interval $[0, 2]$

   (c) (5 pts) Suppose $\mathbf{v}, \mathbf{w} \in V$, a vector space with an inner product $\langle\ ,\ \rangle$. Show that $\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 = 2\left(\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2\right)$.

   (d) (5 pts) Suppose $A = \begin{pmatrix} 4 & -1 \\ -1 & d \end{pmatrix}$. Find all values of d so that $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T A \mathbf{w}$ is a valid inner product on $\mathbb{R}^2$.

Solution

(a)

$$\int_0^2 f(x)g(x)dx \le \sqrt{\int_0^2 f^2(x)dx}\sqrt{\int_0^2 g^2(x)dx}$$

For the example,

$$\langle f, g \rangle = 4 \le \|f\|\|g\| = \sqrt{8/3}\sqrt{32/5} = \frac{16}{\sqrt{15}}.$$

This is valid because $\sqrt{15} < \sqrt{16} = 4$.

(b) $\|x\|_\infty = 2$, $\|x^2\|_\infty = 4$.

(c)

$$\begin{aligned}
\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 &= \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle + \langle \mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w} \rangle \\
&= \|\mathbf{v}\|^2 + 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2 + \|\mathbf{v}\|^2 - 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2 \\
&= 2\left(\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2\right)
\end{aligned}$$

Here we used the definition of norm, and bilinearity and symmetry (which wasn't strictly needed...) of the inner product.

(d) We need A to be positive definite. For the 2×2 case this only requires that $a_{11} = 4 > 0$ and $\det(A) = 4d - 1 > 0$, so $d > \frac{1}{4}$. This can be easily shown using row operations to find the LU decomposition, as well.

3. (15 pts) Let A be a real $m \times n$ matrix with $m \geq n$ and rank n. Let $A = QR$ be the QR factorization obtained using Gram-Schmidt, i.e., with a square R matrix.

   (a) (5 pts) Prove that the Gram matrix $A^T A$ is positive definite. (You may quote any relevant theorem from class.)

   (b) (5 pts) Does $A^T A$ have a positive determinant? Justify your answer.

   (c) (5 pts) Prove that $R^T R$ is the Cholesky factorization of the Gram matrix $A^T A$. You may use the fact that the Cholesky factorization is unique.

Solutions: (a) We know that a Gram matrix is symmetric and positive definite when the columns of A are linearly independent. The problem states that the rank of A equals the number of columns, so the columns must be linearly independent, and the Gram matrix must be positive definite.

(b) Yes. Consider the LDL^T decomposition of a postivie definite matrix. All entries of D are positive and thus the determinant will be positive.

(c) First plug the QR factorization into the definition of the Gram matrix to obtain

$$A^T A = (QR)^T QR = R^T Q^T QR.$$

Now note that the columns of Q are orthonormal which implies that $Q^T Q = I$. This leaves

$$A^T A = R^T R.$$

Note that $R^T R$ is a factorization of the Gram matrix into the product of a lower-triangular matrix R^T with positive diagonal elements and its transpose R.

4. (15 pts)

   (a) (5 pts) Is the set of complex vectors of the form $\begin{pmatrix} z \\ \overline{z} \end{pmatrix}$ a subspace of $\mathbb{C}^2$? (Note that $\overline{z}$ denotes the complex conjugate of z.) Be sure to justify your answer.

   (b) (5 points) Consider the following vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ \mathrm{i} \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1+\mathrm{i} \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -1+\mathrm{i} \\ 1+\mathrm{i} \\ -1 \end{pmatrix}.$$

   Which of these are orthogonal to each other with respect to the Hermitian dot product?

   (c) (5 pts) A complex matrix H is called Hermitian if it equals its Hermitian adjoint, i.e., $H = \overline{H^T}$. Prove that all the diagonal entries of a Hermitian matrix are real.

Solution:

(a) No. The set is not closed under complex multiplication. For a counterexample, let $\mathbf{x} := i(1,1)^T$. It is not true that $x_2 = \overline{x}_1$.

(b) In the Hermitian dot product, $\mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 0$

(c) Consider h_{ii} to be a diagonal element of H. Since we know that $H = \overline{H^T}$, we know that $h_{ii} = \overline{h}_{ii}$. The only way for this to be true is for $h_{ii} \in \mathbb{R}^n$ for all i.

5. (26 pts) Let $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$

   (a) (12 pts) Use Gram-Schmidt to find an orthonormal basis for the image of A.

   (b) (7 pts) What are the orthogonal Q and upper triangular R such that $A = QR$.

   (c) (7 pts) If you were to use Householder transformations to factor A into QR, what would be the unit vector needed to create the first H matrix?

Solution:

(a) We first find $\mathbf{u}_1$ and r_{11}:

$$r_{11} = \sqrt{\mathbf{a}_1^T \mathbf{a}_1} = \sqrt{3}$$

$$\mathbf{u}_1 = \frac{1}{r_{11}} \mathbf{a}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Next we find r_{12} and r_{22} and use them to calculate $\mathbf{u}_2$:

$$r_{12} = \mathbf{u}_1^T \mathbf{a}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$$r_{22} = \sqrt{\mathbf{a}_2^T \mathbf{a}_2 - r_{12}^2} = \sqrt{9 - 3} = \sqrt{6}$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{6}} \left(\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} - \sqrt{3} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right) = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Finally we calculate r_{13}, r_{23}, and r_{33} and use them to calculate $\mathbf{u}_3$:

$$r_{13} = \mathbf{u}_1^T \mathbf{a}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$$r_{23} = \mathbf{u}_2^T \mathbf{a}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \frac{6}{\sqrt{6}} = \sqrt{6}$$

$$r_{33} = \sqrt{\mathbf{a}_3^T \mathbf{a}_3 - r_{13}^2 - r_{23}^2} = \sqrt{11 - 3 - 6} = \sqrt{2}$$

$$\mathbf{u}_3 = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} - \sqrt{3} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \sqrt{6} \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

So our orthonormal basis is

$$\left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(b) The orthonormal basis vectors become the columns of our Q matrix:

$$Q = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ -1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{pmatrix}$$

and the entries of R can be read from the coefficients:

$$R = \begin{pmatrix} \sqrt{3} & \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{6} & \sqrt{6} \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

(c) The first Householder matrix reflects $\mathbf{a}_1$ into the direction of $\mathbf{e}_1$. Since $|\mathbf{a}_1| = \sqrt{3}$ we need to reflect onto $\sqrt{3}\mathbf{e}_1$ and our first unit vector is

$$\mathbf{u} = \frac{\mathbf{a}_1 - \sqrt{3}\mathbf{e}_1}{\mathbf{a}_1 - \sqrt{3}\mathbf{e}_1}$$

$$\mathbf{a}_1 - \sqrt{3}\mathbf{e}_1 = \begin{pmatrix} 1 - \sqrt{3} \\ -1 \\ 1 \end{pmatrix}$$

$$|\mathbf{a}_1 - \sqrt{3}\mathbf{e}_1| = \sqrt{(1-\sqrt{3})^2 + 1 + 1} = \sqrt{1 - 2\sqrt{3} + 3 + 2} = \sqrt{6 - 2\sqrt{3}}$$

$$\mathbf{u} = \frac{1}{\sqrt{6 - 2\sqrt{3}}} \begin{pmatrix} 1 - \sqrt{3} \\ -1 \\ 1 \end{pmatrix}$$