Write your name and your professor's name or your section number below. You are *not* allowed to use textbooks, notes, or a calculator. To receive full credit on a problem you must show **sufficient justification for your conclusion** unless explicitly stated otherwise.

Name:

Instructor and Section:

1. (24 pts) If the statement is **always true**, write "TRUE"; if it is possible for the statement to be false then mark "FALSE." You must give a **justification** for your answer. That is, if the answer is true, provide a brief proof. If the answer is false, provide a counterexample.

   (a) If you reorder the basis vectors before starting the Gram-Schmidt process you will get the same basis.

   (b) If A is non-singular, then both $A^T A$ and AA^T are positive definite.

   (c) The product $\langle f, g\rangle = \displaystyle\int_{-1}^{1} f(x)g(x)x dx$ is a valid inner product on C^0.

   (d) The image of a square matrix is orthogonal to its kernel when using the dot product.

2. (20 pts) The following questions are unrelated.

   (a) (5 pts) Write out explicitly the Cauchy-Schwartz inequality for the the L^2 inner product on the interval $[0, 2]$. Show that it is valid for the functions $f(x) = x$ and $g(x) = x^2$.

   (b) (5 pts) Compute the L^∞ norms of the functions f and g from (a) on the interval $[0, 2]$

   (c) (5 pts) Suppose $\mathbf{v}, \mathbf{w} \in V$, a vector space with an inner product $\langle \ , \ \rangle$. Show that $\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 = 2\left(\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2\right)$.

   (d) (5 pts) Suppose $A = \begin{pmatrix} 4 & -1 \\ -1 & d \end{pmatrix}$. Find all values of d so that $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T A \mathbf{w}$ is a valid inner product on $\mathbb{R}^2$.

3. (15 pts) Let A be a real $m \times n$ matrix with $m \geq n$ and rank n. Let $A = QR$ be the QR factorization obtained using Gram-Schmidt, i.e., with a square R matrix.

   (a) (5 pts) Prove that the Gram matrix $A^T A$ is positive definite. (You may quote any relevant theorem from class.)

   (b) (5 pts) Does $A^T A$ have a positive determinant? Justify your answer.

   (c) (5 pts) Prove that $R^T R$ is the Cholesky factorization of the Gram matrix $A^T A$. You may use the fact that the Cholesky factorization is unique.

4. (15 pts)

   (a) (5 pts) Is the set of complex vectors of the form $\begin{pmatrix} z \\ \overline{z} \end{pmatrix}$ a subspace of $\mathbb{C}^2$? (Note that $\overline{z}$ denotes the complex conjugate of z.) Be sure to justify your answer.

   (b) (5 points) Consider the following vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ \mathrm{i} \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1+\mathrm{i} \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -1+\mathrm{i} \\ 1+\mathrm{i} \\ -1 \end{pmatrix}.$$

   Which of these are orthogonal to each other with respect to the Hermitian dot product?

   (c) (5 pts) A complex matrix H is called Hermitian if it equals its Hermitian adjoint, i.e., $H = \overline{H^T}$. Prove that all the diagonal entries of a Hermitian matrix are real.

5. (26 pts) Let $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$

   (a) (12 pts) Use Gram-Schmidt to find an orthonormal basis for the image of A.

   (b) (7 pts) What are the orthogonal Q and upper triangular R such that $A = QR$.

   (c) (7 pts) If you were to use Householder transformations to factor A into QR, what would be the unit vector needed to create the first H matrix?