Write your name and your professor's name or your section number below. You are *not* allowed to use textbooks, or a calculator. You may have two 3x5" cards (both sides) for notes, or equivalent. To receive full credit on a problem you must show **sufficient justification for your conclusion** unless explicitly stated otherwise.

Name:

Instructor and Section: Mitchell, 002

1. (30 pts) If the statement is **always true**, write "TRUE"; if it is possible for the statement to be false then mark "FALSE." You must give a **justification** for your answer. That is, if the answer is true, provide a brief proof. If the answer is false, provide a counterexample.

   (a) A singular matrix is not diagonalizable.

   (b) If the incomplete matrix A has a Jordan Canonical Form (JCF) J, then the JCF of A^2 is J^2.

   (c) The quadratic function $p(x, y) = 2x^2 - 2xy + y^2 - 2x + 4y + 7$ has a minimum value.

   (d) If a matrix is not symmetric, then it is incomplete.

   (e) If a system of equations $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions then the matrix A is singular.

Solution

(a) **False.** A counterexample is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

which is singular and already diagonalized, since $S = I$

(b) **False.** The square of a Jordan block is generally not another Jordan block, and so cannot be in the Jordan Canonical Form of the squared matrix. For example, if

$$J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

then

$$J^2 = \begin{pmatrix} 4 & 4 \\ 0 & 4 \end{pmatrix}$$

which is not a Jordan block since the entry on the superdiagonal is not 1.

(c) **True.** Here we have a positive definite matrix:

$$K = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & -1 \\ 0 & 1/2 \end{pmatrix}$$

so $p(x, y)$ will have a minimum value.

(d) **False** A counterexample would be

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

which is not symmetric, but is still complete.

(e) **False** This is true when A is square, but for a nonsquare system there can be infinitely many solutions whenever $rank(A) < n$.

2. (20 pts) The following questions are unrelated.

   (a) (8 pts) Is $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \\ -x \end{pmatrix}$ a linear function? If it is linear, prove that it is and find the matrix representation of L in the standard basis.

   (b) (12 pts) Let L be the linear function with the standard basis representation given by $A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$. Find bases for $\mathbb{R}^2$ and $\mathbb{R}^3$ that puts L into canonical form after a change of basis.

Solution

(a) Yes, this is a linear function, since

$$L(a\mathbf{x} + b\mathbf{y}) = \begin{pmatrix} ax_2 + by_2 \\ 0 \\ -ax_1 - by_1 \end{pmatrix} = aL(\mathbf{x}) + bL(\mathbf{y})$$

The matrix representation has columns

$$L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \qquad L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \end{pmatrix}$$

(b) For $\mathbb{R}^3$, we need basis vectors for the coimage and kernel. Row reducing A allows us to find the kernel:

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 2 \end{pmatrix} \xrightarrow[R_2 = R_2 - R_1]{} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \end{pmatrix}$$
$$\xrightarrow[R_1 = R_1 + R_2]{} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

So the basis vector for the kernel is

$$z = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

We take the transposes of the rows of the REF as a basis for the cokernel, so our basis is

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \right\}$$

To find the $\mathbb{R}^2$ basis, we apply the linear transformation to the coimage basis:

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

So the $\mathbb{R}^2$ basis is

$$\left\{ \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \end{pmatrix} \right\}$$

3. (20 pts) Let $A = \begin{pmatrix} -1 & 2 \\ -4 & 5 \end{pmatrix}$.

   (a) (8 pts) What is the characteristic polynomial $p(\lambda)$ for A?

   (b) (8 pts) Find the eigenvalues and eigenvectors for A.

   (c) (4 pts) Is A a complete matrix? What is the diagonalization transformation for A? (The Λ matrix in $S\Lambda S^{-1}$)

Solution:

(a) $p(\lambda) = \lambda^2 - 4\lambda + 3$

(b) $\lambda_1 = 3$, $\mathbf{v}_1 = (1,2)^T$ $\lambda_2 = 1$, $\mathbf{v}_2 = (1,1)^T$

(c) Yes. Set $P = (\mathbf{v}_1, \mathbf{v}_2)$, then $A = P\Lambda P^{-1}$, where $\Lambda = diag(3,1)$.

4. (20 pts) Let A be a matrix with characteristic polynomial given by

$$p_A(\lambda) = (1-\lambda)^3(2-\lambda)^3(-3-\lambda)$$

Eigenvalue $\lambda = 1$ has one ordinary eigenvector while eigenvalue $\lambda = 2$ has two ordinary eigenvectors.

(a) (16 pts) Write down all of the Jordan blocks that appear in A's Jordan Canonical Form.

(b) (4 pts) What is the dimension the of eigenspace for eigenvalue 2?

Solution:

(a) Eigenvalue $\lambda = 1$ has algebraic multiplicity 3 and geometric multiplicity 1, so it will have one 3 Jordan block:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Eigenvalue $\lambda = 2$ has algebraic multiplicity 3 and geometric multiplicity 2, so it will have one 2 Jordan block and one 1×1:

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 \end{pmatrix}$$

Finally , eigenvalue $\lambda = -3$ has algebraic multiplicity 1 and geometric multiplicity 1, so it will have one 1 Jordan block:

$$\begin{pmatrix} -3 \end{pmatrix}$$

(b) Since $\lambda = 2$ has two ordinary eigenvectors, the dimension of the eigenspace is 2.

5. (20 pts) Let A be the matrix with the SVD given by

$$A = \begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{7} \\ 1/\sqrt{3} & 2/\sqrt{7} \\ 1/\sqrt{3} & -1/\sqrt{7} \\ 0 & 1/\sqrt{7} \end{pmatrix} \begin{pmatrix} 4\sqrt{15} & 0 \\ 0 & \sqrt{35} \end{pmatrix} \begin{pmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$$

(a) (2 pts) What is the rank of A? Be sure to justify your answer.

(b) (6 pts) Does $A\mathbf{x} = \mathbf{b}$ have a solution when $\mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$?

(c) (12 pts) What is the best rank 1 approximation of A using the Froebenius norm?

Solution:

(a) A has two singular values and so it's rank is 2.

(b) Yes. $\mathbf{b}$ is a multiple of the first column of P. Since the columns of P are a basis for the image of the matrix, $A\mathbf{x} = \mathbf{b}$ has a solution.

(c) We use the largest singular value and its singular vectors to construct the rank 1 approximation:

$$\begin{aligned} A_1 &= \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \end{pmatrix} \left(4\sqrt{15}\right) \begin{pmatrix} -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \\ &= \frac{4\sqrt{15}}{\sqrt{3}\sqrt{5}} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 & 2 \end{pmatrix} \\ &= 4 \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ -1 & 2 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 4 & -8 \\ -4 & 8 \\ -4 & 8 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

6. (20 pts) Let B be the matrix with the SVD given by

$$B = \begin{pmatrix} -1/2 & -1/2 \\ 1/2 & -1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(a) (8 pts) Find the psuedoinverse of B.

(b) (4 pts) Find the least squares solution to $B\mathbf{x} = \mathbf{c}$ when $\mathbf{c} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$.

(c) (8 pts) What is the closest point to $\mathbf{c}$ in img$B$?

Solution:

(a) The pseudoinverse of B is given by

$$\begin{aligned} B^+ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/4 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} -1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1/8 & 1/8 & 1/8 & 1/8 \\ -1/4 & -1/4 & -1/4 & 1/4 \end{pmatrix} \\ &= \begin{pmatrix} -1/4 & -1/4 & -1/4 & 1/4 \\ -1/8 & 1/8 & 1/8 & 1/8 \end{pmatrix} \end{aligned}$$

(b) The least squares solution is $B^+\mathbf{c}$:

$$\mathbf{x}^* = \begin{pmatrix} -1/4 & -1/4 & -1/4 & 1/4 \\ -1/8 & 1/8 & 1/8 & 1/8 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/4 \\ 3/8 \end{pmatrix}$$

(c) The closest point in img$B$ is $B\mathbf{x}^*$:

$$\begin{aligned}
\mathbf{w} &= \begin{pmatrix} -1/2 & -1/2 \\ 1/2 & -1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1/4 \\ 3/8 \end{pmatrix} \\
&= \begin{pmatrix} -1/2 & -1/2 \\ 1/2 & -1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3/8 \\ -1/4 \end{pmatrix} \\
&= \begin{pmatrix} -1/2 & -1/2 \\ 1/2 & -1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix} \\
&= \begin{pmatrix} -1/2 \\ 1 \\ 1 \\ 1/2 \end{pmatrix}
\end{aligned}$$

7. (20 points) The 3×3 matrix A has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 3$ and $\lambda_3 = 4$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, respectively.

   (a) (4 pts) Does this uniquely define the matrix A? Explain your answer.

   (b) (8 pts) Find a matrix A corresponding to the given eigenvalues/eigenvectors (either "a" matrix, or "the" matrix, depending on your answer to the previous question).

   (c) (8 pts) Find e^{tA}.

Solution:

(a) Yes, this is unique, since the eigenvectors are linearly independent (they *have* to be, since they correspond to different eigenvalues), so they form a basis for $\mathbb{R}^3$, so we've completely specified the linear operator, and its representation as a matrix (i.e., with the canonical basis) is also unique.

If we make $S = (\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3)$ and $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ so that $A = S\Lambda S^{-1}$, you might wonder what if you defined $S = (\mathbf{v}_3 \mathbf{v}_2 \mathbf{v}_1)$ or some other permutation. As long as you permuted Λ and used the new S^{-1}, then you'd get the same matrix A.

Similarly, if $\mathbf{v}_1 = [2\ 0\ 0]^T$ is an eigenvector, then $[1\ 0\ 0]^T$ is also an eigenvector, and you could use that just as well, and it wouldn't change A.

(b) The matrix A is $A = S\Lambda S^{-1}$ where S and Λ are as above, so

$$S = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Finding S^{-1} by Gauss-Jordan gives

$$S^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

And so our A matrix is

$$
\begin{aligned}
A &= S\Lambda S^{-1} \\
&= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 3 & 0 \\ 4 & 0 & -4 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 3 \end{pmatrix}
\end{aligned}
$$

(c) Since the matrix is diagonalizable, we can define $e^{tA} = Se^{t\Lambda}S^{-1}$ and the exponential of a diagonal matrix is done entrywise to the diagonal entries. Hence:

$$
\begin{aligned}
e^t At &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{4t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 & e^{2t} \\ 0 & e^{3t} & 0 \\ e^{4t} & 0 & -e^{4t} \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} e^{2t} + e^{4t} & 0 & e^{2t} - e^{4t} \\ 0 & 2e^{3t} & 0 \\ e^{2t} - e^{4t} & 0 & e^{2t} + e^{4t} \end{pmatrix}
\end{aligned}
$$