APPM 3570/STAT 3100 — Exam 3 — Fall 2025

On the front of your bluebook, write (1) **your name**, (2) **Exam 3**, (3) **APPM 3570/STAT 3100**. Correct answers with no supporting work may receive little or no credit. Books, notes and electronic devices of any kind are not allowed. Your exam should be uploaded to Gradescope in a PDF format (Recommended: **Genius Scan**, **Scannable** or **CamScanner** for iOS/Android). **Show all work, justify your answers. Do all problems.** Students are required to re-write the **honor code statement** in the box below on the **first page** of their exam submission and **sign and date it**:

On my honor, as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work. Signature:____________________________ Date:_________

1. [EXAM03] (30pts) There are 3 unrelated parts to this question. Justify your answers.

   (a) (10pts) Suppose that A and B each randomly and independently choose 3 of 10 objects. Find the expected number of objects chosen exclusively by A or B (but not both).

   (b) (10pts) A fair die is successively rolled. Let X and Y denote, respectively, the number of rolls necessary to obtain the first 6 and the first 5. Find $P(X = 2|Y = 5)$.

   (c) (10pts) If X_1, X_2, X_3, and X_4 are (pairwise) uncorrelated random variables, each having mean 0 and variance 1, compute the correlation of $X_1 + X_2$ and $X_2 + X_3$, that is, find $\rho(X_1 + X_2, X_2 + X_3)$.

Solution:

(a)(10pts) Let $X_i = \begin{cases} 1, & \text{if } A \text{ or } B \text{ choose item } i \text{ but not both,} \\ 0, & \text{else,} \end{cases}$ and let $X = \sum_{i=1}^{10} X_i$ and note that

$$P(X_i = 1) = P(\{{}^{A \text{ chooses item } i}_{\text{and } B \text{ doesn't}}\} \text{ or } \{{}^{B \text{ chooses item } i}_{\text{and } A \text{ doesn't}}\}) = \frac{3}{10} \cdot \frac{7}{10} + \frac{7}{10} \cdot \frac{3}{10} = 2 \cdot \frac{3}{10} \cdot \frac{7}{10}$$

and so the expected number of objects chosen exclusively by A or B (but not both) is

$$E[X] = \sum_{i=1}^{10} E[X_i] = \sum_{i=1}^{10} P(X_i = 1) = \sum_{i=1}^{10} 2\left(\frac{3}{10}\right)\left(\frac{7}{10}\right) = \boxed{10 \cdot 2\left(\frac{3}{10}\right)\left(\frac{7}{10}\right)} = 4.2.$$

(b)(10pts) Proceeding by definition, we have

$$P(X = 2|Y = 5) = \frac{P(X = 2, Y = 5)}{P(Y = 5)} = \frac{\frac{4}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6}}{\left(\frac{5}{6}\right)^4 \frac{1}{6}} = \frac{\frac{4}{6}\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)^2}{\left(\frac{5}{6}\right)^4 \frac{1}{6}} = \boxed{\frac{4}{25}} = 0.16.$$

(c)(10pts) Since the random variables are uncorrelated we have $\operatorname{Cov}(X_i, X_j) = 0$ for $i \neq j$, and so

$$\begin{aligned} \operatorname{Cov}(X_1 + X_2, X_2 + X_3) &= \operatorname{Cov}(X_1, X_2 + X_3) + \operatorname{Cov}(X_2, X_2 + X_3) \\ &= \underbrace{\operatorname{Cov}(X_1, X_2)}_{=0} + \underbrace{\operatorname{Cov}(X_1, X_3)}_{=0} + \operatorname{Cov}(X_2, X_2) + \underbrace{\operatorname{Cov}(X_2, X_3)}_{=0} = \operatorname{Var}(X_2) = 1. \end{aligned}$$

For the variance, we have

$$\begin{aligned} \operatorname{Var}(X_1 + X_2) &= \operatorname{Cov}(X_1 + X_2, X_1 + X_2) \\ &= \operatorname{Cov}(X_1, X_1) + 2\operatorname{Cov}(X_1, X_2) + \operatorname{Cov}(X_2, X_2) = \operatorname{Var}(X_1) + \operatorname{Var}(X_2) = 2 \end{aligned}$$

and, similarly, $\operatorname{Var}(X_2 + X_3) = 2$. Thus the correlation of $X_1 + X_2$ and $X_2 + X_3$ is

$$\rho(X_1 + X_2, X_2 + X_3) = \frac{\operatorname{Cov}(X_1 + X_2, X_2 + X_3)}{\sqrt{\operatorname{Var}(X_1 + X_2)}\sqrt{\operatorname{Var}(X_2 + X_3)}} = \frac{\operatorname{Var}(X_2)}{\sqrt{2} \cdot \sqrt{2}} = \boxed{\frac{1}{2}}$$

2. [Exam03] (40pts) Let X and Y have the joint density function

$$f_{X,Y}(x,y) = \begin{cases} 2(xy+y^2), & \text{if } 0<y<1,\ -y<x<y, \\ 0, & \text{otherwise.} \end{cases}$$

where $E[X|Y] = y/3$ and the marginal pdf of X is $f_X(x) = \begin{cases} 2/3 + x - 5x^3/3, & \text{if } x \in (-1,0) \\ 2/3 + x - x^3/3, & \text{if } x \in [0,1) \\ 0, & \text{else} \end{cases}$

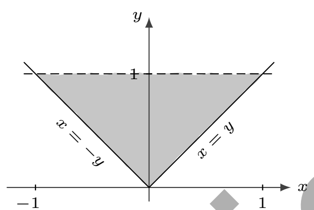
(a) (10pts) Set-up the integral(s) we would need to calculate to find $P(|X| < 0.4, |Y| < 0.4)$. (Set-up only!)

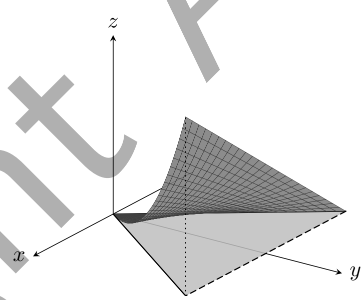
(b) (10pts) Find the marginal pdf $f_Y(y)$ and the conditional pdf $f_{X|Y}(x\,|\,0.4)$. (Be sure to explicitly specify the domains as appropriate.)

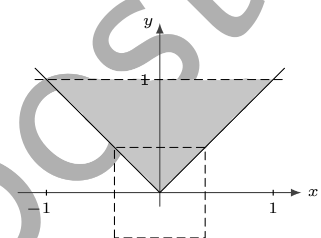
(c) (10pts) Find the expectation $E[X]$.

(d) (10pts) Find the conditional probability $P(0 < Y < 0.5\,|\,X = 0)$.

Solution:







(a)(10pts) Note that $\{|X| < 0.4, |Y| < 0.4\} = \{-0.4 < X < 0.4, -0.4 < Y < 0.4\}$, so

$$P(|X| < 0.4, |Y| < 0.4) = P(-Y < X < Y, 0 < Y < 0.4) = \boxed{\int_0^{0.4}\int_{-y}^{y} 2(y^2+xy)\,dx\,dy}$$

or

$$P(|X| < 0.4, |Y| < 0.4) = \int_{-0.4}^{0}\int_{-x}^{0.4} 2(y^2+xy)\,dy\,dx + \int_0^{0.4}\int_x^{0.4} 2(y^2+xy)\,dy\,dx.$$

(b)(10pts) To find the marginal pdf $f_Y(y)$, note that, for each $y \in (0,1)$, we have

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{-y}^{y} 2(y^2+xy)\,dx = 2\Big(xy^2 + \frac{x^2y}{2}\Big)\Big|_{-y}^{y} = 4y^3.$$

$$\Rightarrow \boxed{f_Y(y) = 4y^3 \text{ for } 0<y<1 \text{ and } 0 \text{ else.}}$$

If $y = 0.4$, then $-0.4 < x < 0.4$, so the conditional pdf of $X|Y = 0.4$ is

$$\boxed{f_{X|Y}(x|0.4) = \frac{f(x,0.4)}{f_Y(0.4)} = \begin{cases} \dfrac{2\cdot 0.4\cdot(x+0.4)}{4(0.4)^3}, & \text{for } -0.4<x<0.4, \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} \dfrac{x+0.4}{2(0.4)^2}, & \text{for } -0.4<x<0.4, \\ 0, & \text{otherwise.} \end{cases}}$$

(c)(10pts) Using the formula $E[X] = E\big[E[X|Y]\big]$, we have

$$E[X] = E\big[E[X|Y]\big] = \int_{-\infty}^{\infty} E[X|Y] f_Y(y)\,dy = \int_0^1 \frac{y}{3}\cdot 4y^3\,dy = \frac{4}{3}\cdot\frac{y^5}{5}\Big|_0^1 = \boxed{\frac{4}{15}}$$

(d)(10pts) To find $P(0 < Y < 0.5 \,|\, X = 0)$, we need the conditional pdf $f_{Y|X}(y\,|\,0)$. Note that $f_X(0) = 2/3$ and, if $x = 0$, then $0 < y < 1$, so the pdf of $Y|X = 0$ is

$$f_{Y|X}(y\,|\,0) = \frac{f(0,y)}{f_X(0)} = \begin{cases} \dfrac{2(0\cdot y + y^2)}{2/3}, & \text{if } 0 < y < 1, \\ 0, & \text{otherwise,} \end{cases} = \begin{cases} 3y^2, & \text{if } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Now, to find the probability $P(0 < Y < 0.5 \,|\, X = 0)$, we have

$$P(0 < Y < 0.5 | X = 0) \int_0^{1/2} f_{Y|X}(y\,|\,0)\,dy = \int_0^{1/2} 3y^2\,dy = y^3\Big|_0^{1/2} = \boxed{\frac{1}{8}}$$

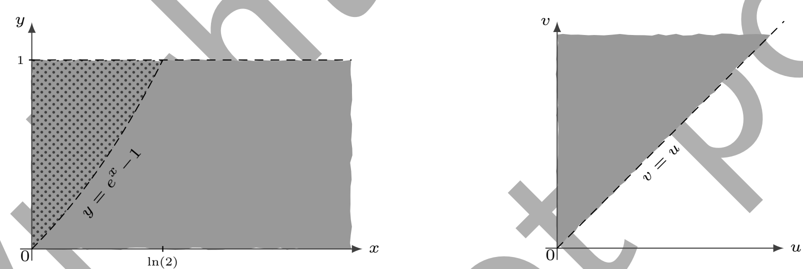
3. [EXAM03] (30pts) Suppose X and Y are independent random variables, where X is Exponentially distributed with $\lambda = 1$ and Y is Uniformly distributed on $(0, 1)$.

   (a) (10pts) Set-up the integral(s) we would need to calculate to find $P(Y > e^X - 1)$. (Set-up only!)

   (b) (10pts) For $X > 0$ and $Y > 0$ define $U = X, V = \dfrac{X}{Y}$. Find the joint pdf $f_{U,V}(u, v)$. (The pdf should be defined for all of $\mathbb{R}^2$.)

   (c) (10pts) Find the conditional expectation $E[V^{-1} \,|\, U]$.

Solution:



(a) xy-domain (b) uv-domain

(a)(10pts) By independence, the joint pdf is $f_{X,Y}(x,y) = f_X(x)f_Y(y) = 1 \cdot e^{-x} = e^{-x}$, $x > 0$, $0 < y < 1$ and 0 otherwise. Thus,

$$P(Y > e^X - 1) = P(0 < X < \ln(2),\, e^x - 1 < Y < 1) = \boxed{\int_0^{\ln(2)} \int_{e^x - 1}^{1} e^{-x}\,dydx}$$

(b)(10pts) Suppose $X \sim \text{Exponential}(\lambda = 1)$ and $Y \sim \text{Uniform}(0,1)$ and X, Y are independent. Note, if $U = X$ and $V = X/Y$, then $X = U$ and $Y = X/V = U/V$ and

$$J(x,y) = \begin{vmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1/y & -x/y^2 \end{vmatrix} = -x/y^2 \Rightarrow |J(x,y)|^{-1} = \frac{y^2}{x} = \frac{(u/v)^2}{u} = \frac{u}{v^2} = |J(u,v)|$$

and note that $x > 0$ implies $u > 0$ and $0 < y < 1$ implies $0 < \frac{u}{v} < 1$ and, since $v = \frac{x}{y} > 0$, we can conclude that $0 < u < v$. Since $f_{X,Y}(x,y) = 1 \cdot e^{-x}$ if $x > 0, 0 < y < 1$ and 0 else, we have

$$\boxed{f_{U,V}(u,v) = f_{X,Y}(u, u/v) \cdot |J(x,y)|^{-1} = \begin{cases} e^{-u} \cdot \dfrac{u}{v^2}, & \text{for } u > 0,\, v > u, \\ 0, & \text{else.} \end{cases}}$$

(c)(10pts) Since $U = X$ and $X \sim \text{Exponential}(\lambda = 1)$, we have $f_U(u) = e^{-u}$ for $u > 0$ and 0 otherwise, thus, for each $u > 0$, we have

$$f_{V|U}(v \mid u) = \frac{f(u,v)}{f_U(u)} = \begin{cases} \dfrac{u}{v^2}, & \text{if } v > u, \\ 0, & \text{else.} \end{cases}$$

Thus, for each $u > 0$, we have

$$E[V^{-1} \mid U] = \int_{-\infty}^{\infty} \frac{1}{v} \cdot f_{V|U}(v|u)\, dv = \int_u^{\infty} \frac{1}{v} \cdot \frac{u}{v^2}\, dv = \int_u^{\infty} uv^{-3}\, dv = -\frac{u}{2v^2}\Big|_u^{\infty} = \frac{u}{2u^2} = \boxed{\frac{1}{2u}} \text{ for } u > 0.$$