1. (27 points) The following problems are unrelated.

   (a) Find the derivative of $f(x) = 7x^2 \cos(\tan x)$.

   (b) Evaluate $\displaystyle\int \frac{x^2}{(x^3+4)^2}\,dx$.

   (c) Evaluate $\displaystyle\lim_{x\to 0} \frac{\arcsin(2x)}{5x}$.

   Solution:

   (a)
   $$f'(x) = -7x^2 \sin(\tan x)\sec^2 x + 14x\cos(\tan x)$$

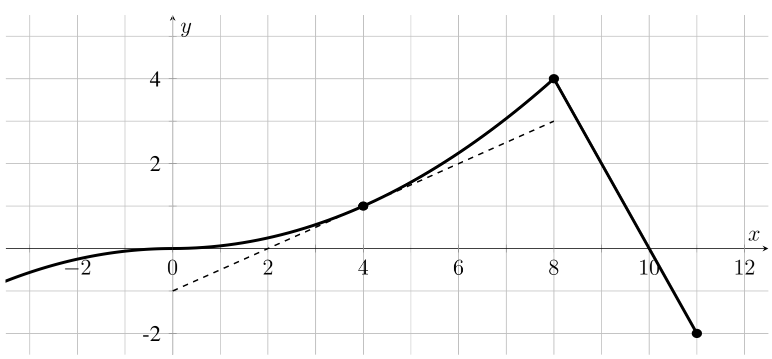
   (b) We will apply the substitution $u = x^3 + 4$. This yields $du = 3x^2 dx$, or $\frac{1}{3}\,du = x^2\,dx$:

   $$\begin{aligned}\int \frac{x^2}{(x^3+4)^2}\,dx &= \frac{1}{3}\int \frac{1}{u^2}\,du \\ &= \frac{1}{3}\cdot\frac{-1}{u} + C \\ &= -\frac{1}{3(x^3+4)} + C.\end{aligned}$$

   (c) We note that the given limit is a $\frac{0}{0}$-indeterminate form, so we can apply L'Hospital's Rule:

   $$\lim_{x\to 0} \underbrace{\frac{\arcsin(2x)}{5x}}_{0/0} \overset{LH}{=} \lim_{x\to 0} \frac{\frac{2}{\sqrt{1-4x^2}}}{5} = \frac{2}{5}$$

2. (32 points) Consider the <u>odd</u> function $f(x)$ defined on $[-11, 11]$. A <u>portion</u> of its graph is shown below. The dashed line corresponds to the line tangent to $f(x)$ at $x = 4$.



   (a) Evaluate $\displaystyle\int_{-8}^{11} f(x)\,dx$.

   (b) Find the linearization of $f(x)$ at $a = -4$.

(c) Let $g(x) = \int_{-11}^{x} f(t)\,dt$. Find $g'(9)$.

(d) Evaluate $\lim_{x\to 4} \frac{f(x)-1}{6x-24}$.

Solution:

(a) Because f is odd, $\int_{-8}^{8} f(x)\,dx = 0$. Therefore

$$\int_{-8}^{11} f(x)\,dx = \int_{8}^{11} f(x)\,dx = \int_{8}^{10} f(x)\,dx + \int_{10}^{11} f(x)\,dx = \frac{1}{2}(2\cdot 4) - \frac{1}{2}(1\cdot 2) = 3,$$

subtracting the areas of two triangles.

(b) Because f is odd, $f(-4) = -f(4) = -1$ and $f'(-4) = f'(4) = \frac{1}{2}$. Therefore the linearization of f at $x = -4$ is

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= f(-4) + f'(-4)(x+4) \\ &= -1 + \frac{1}{2}(x+4). \end{aligned}$$

(c) By FTC-1, $g'(x) = \frac{d}{dx}\int_{-11}^{x} f(t)\,dt = f(x)$, so $g'(9) = f(9) = 2$.

(d) Note that $f(4) = 1$, so the given limit is a $\frac{0}{0}$-indeterminate form. We can apply L'Hospital's Rule:

$$\lim_{x\to 4} \underbrace{\frac{f(x)-1}{6x-24}}_{0/0} \overset{LH}{=} \lim_{x\to 4} \frac{f'(x)}{6} = \frac{f'(4)}{6} = \frac{1/2}{6} = \frac{1}{12}.$$

3. (13 points) In the year 1990, the population of Tralfamador was 5 billion. By the year 2010, the population was 6 billion. Assuming the growth of the population is proportional to the current size of the population, that is $\frac{dP}{dt} = kP$, what will the population be in the year 2042?

Solution:

We will model the population with $P(t) = P_0 e^{kt}$ where t measures years since 1990 and $P(t)$ measures billions of tralfamadorians. We have $P(0) = 5$, $P(20) = 6$, and we want to know $P(52)$. Note that $P_0 = 5$. We can find k with the second data point:

$$\begin{aligned} P(20) &= 6 \\ 5e^{20k} &= 6 \\ 20k &= \ln\frac{6}{5} \\ k &= \frac{1}{20}\ln\frac{6}{5}. \end{aligned}$$

So, the population in 2042 is expected to be

$$P(52) = 5e^{(52/20)\ln(6/5)} = 5e^{(13/5)\ln(6/5)} = 5\left(\tfrac{6}{5}\right)^{13/5} \text{ billion.}$$

4. (18 points) The graph of a function $f(x)$ is shown below. Suppose $f(x)$ is the **<u>derivative</u>** of $F(x)$. That is, $f(x) = F'(x)$. Assume that $F(x)$ is <u>continuous</u> on the interval $[m, r]$. No justification is required for the following questions. If the answer to any question is "none", write "none".

   (a) On what open interval(s) is F decreasing?

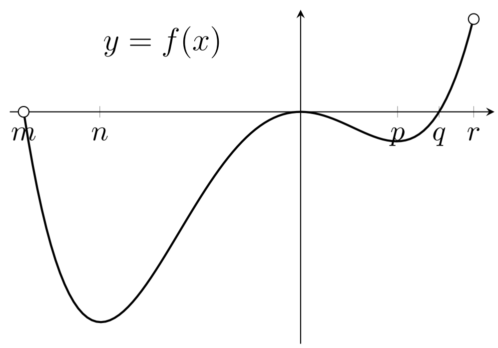
   (b) On what open interval(s) is F concave down?

   (c) What is the x-coordinate of the absolute maximum value of F?

   (d) What is the x-coordinate of the absolute minimum value of F?

   (e) What are the x-coordinates of the inflection point(s) of F?

   (f) At what value(s) of x does $\lim_{h \to 0} \dfrac{F(x+h) - F(x)}{h}$ equal 0?



Solution:

   (a) F is decreasing where $F' = f < 0$ on $(m, 0) \cup (0, q)$.

   (b) F is concave down where $F'' = f' < 0$ (that is, where the slope of f is negative) on $(m, n) \cup (0, p)$.

   (c) By the Extreme Value Theorem, because F is continuous on the closed interval $[m, r]$, it must attain absolute extreme values in the interval. The derivative graph shows that $F(x)$ decreases on almost the entire interval, so the absolute maximum occurs at the left endpoint, $x = m$.

   (d) Because F decreases on $(m, 0) \cup (0, q)$ and increases on (q, r), it attains the absolute minimum at $x = q$.

   (e) F has inflection points where its concavity (and the slope of f) changes at $x = n$, 0, and p.

   (f) The given limit equals $F'(x) = f(x)$ and has the value 0 at $x = 0$ and $x = q$.

5. (19 points) The following problems are unrelated.

   (a) Find the derivative of $y = \left(x^2 + 9x^4\right)^{\sin x}$. Your final answer should be in terms of x, but otherwise unsimplified.

   (b) Consider the integral $\displaystyle\int_1^2 \frac{1}{x}\, dx$.

      i. Approximate this integral using a left Riemann sum (use lefthand endpoints) with three rectangles of equal width. Fully simplify your final answer.

      ii. Is your approximation from (i) an overestimate or underestimate? Provide a *brief* justification for your answer.

Solution:

(a) We will apply Logarithmic Differentiation:

$$\begin{aligned}
\ln y &= \ln\left(x^2+9x^4\right)^{\sin x}\\
\ln y &= \sin x \ln\left(x^2+9x^4\right)\\
\frac{d}{dx}(\ln y) &= \frac{d}{dx}\left(\sin x \ln\left(x^2+9x^4\right)\right)\\
\frac{y'}{y} &= \frac{\sin x(2x+36x^3)}{x^2+9x^4} + \cos x \ln(x^2+9x^4)\\
y' &= \left(x^2+9x^4\right)^{\sin x}\left(\frac{\sin x(2x+36x^3)}{x^2+9x^4} + \cos x \ln(x^2+9x^4)\right).
\end{aligned}$$

(b) i.

$$\int_1^2 \frac{1}{x}\,dx \approx \left[1+\frac{3}{4}+\frac{3}{5}\right]\cdot\frac{1}{3} = \frac{20+15+12}{20}\cdot\frac{1}{3} = \frac{47}{60}.$$

ii. This is an **overestimate** because $y=\frac{1}{x}$ is a decreasing function and we used a left Riemann sum.

6. (27 points) Consider $h(x) = \dfrac{e^x}{1+e^{2x}}$. This function will be used throughout this problem.

(a) Determine $h(\ln 2)$. Be sure to fully simplify your final answer.

(b) Evaluate $\displaystyle\int_0^{\ln\sqrt{3}} h(x)\,dx$. Be sure to fully simplify your final answer.

(c) Use the definition of an even function to show that $h(x)$ is an even function.

Solution:

(a)

$$h(\ln 2) = \frac{e^{\ln 2}}{1+e^{2\ln 2}} = \frac{2}{1+e^{\ln 4}} = \frac{2}{1+4} = \frac{2}{5}.$$

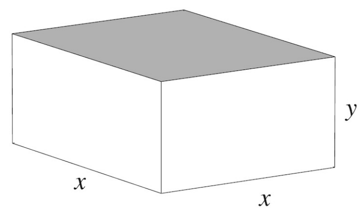
(b) We will use the substitution $u = e^x$. Note that $du = e^x\,dx$.

$$\begin{aligned}
\int_0^{\ln\sqrt{3}} h(x)\,dx &= \int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx\\
&= \int_1^{\sqrt{3}} \frac{1}{1+u^2}\,du\\
&= \arctan(\sqrt{3}) - \arctan(1)\\
&= \frac{\pi}{3} - \frac{\pi}{4}\\
&= \frac{\pi}{12}.
\end{aligned}$$

(c)

$$\begin{aligned}
h(-x) &= \frac{e^{-x}}{1+e^{-2x}}\\
&= \frac{e^{-x}}{1+e^{-2x}}\cdot\frac{e^{2x}}{e^{2x}}\\
&= \frac{e^x}{e^{2x}+1}\\
&= h(x).
\end{aligned}$$

7. (14 points) Irwin is building a cage for his pet snake. The cage will have a volume of 6 cubic feet and a square base measuring x by x feet. Material for the cage top and bottom will cost $\$2$ per square foot, and material for the glass sides will cost $\$3$ per square foot. What value of x will minimize the cost of materials? Use the Second Derivative Test to confirm that x produces a minimum.



Solution:

The cage top and bottom will have a surface area of $2x^2$ sq ft and a cost of $\$2$ per sq ft. The glass sides will have a surface area of $4xy$ sq ft and a cost of $\$3$ per sq ft. The total cost of materials in dollars is

$$\begin{aligned} C &= 2(2x^2) + 3(4xy) \\ &= 4x^2 + 12xy. \end{aligned}$$

The cage volume will be

$$V = x^2 y = 6, \quad \text{so } y = \frac{6}{x^2}.$$

Substituting into the cost function gives

$$\begin{aligned} C &= 4x^2 + 12x \cdot \frac{6}{x^2} \\ &= 4x^2 + \frac{72}{x} \end{aligned}$$

for x in $(0, \infty)$. Next differentiate C.

$$C' = 8x - \frac{72}{x^2}$$

Solve $C' = 0$ to find the critical number(s).

$$\begin{aligned} 0 &= 8x - \frac{72}{x^2} \\ 8x &= \frac{72}{x^2} \\ 8x^3 &= 72 \\ x &= \sqrt[3]{9} \end{aligned}$$

which is the only critical number. The second derivative of C is

$$C'' = 8 + \frac{144}{x^3},$$

which is positive for all $x > 0$, so C is concave up at the critical point and $x = \sqrt[3]{9}$ corresponds to the minimum value. Therefore a square base measuring $\sqrt[3]{9}$ ft by $\sqrt[3]{9}$ ft will minimize the cost of materials.