1. (26 pts) The position function of a particle is given by $s(t) = -t^3 + 3t$ on the interval $[0, 3]$, where t is measured in seconds and s is measured in meters.

(a) i. Find all critical numbers of $s(t)$ on the interval $[0, 3]$. If none exist, clearly state "none".

Solution:

$$s'(t) = -3t^2 + 3$$

Critical numbers are values of t on the specified interval such that $s'(t) = 0$ or $s'(t)$ does not exist. There are no critical numbers of the latter type for this function.

$$\begin{aligned} s'(t) = -3t^2 + 3 &= 0 \\ 3(1 - t^2) &= 0 \\ (1 - t)(1 + t) &= 0 \\ t &= \pm 1 \end{aligned}$$

$\boxed{t = 1}$ is the only critical number of this function that lies on the interval $[0, 3]$.

ii. Identify the absolute maximum and minimum values of $s(t)$ on the given interval and the corresponding values of t at which they occur.

Solution:

The Closed Interval Method compares the function values at the critical numbers and the boundaries of the interval. For this function, the interval boundaries are at $t = 0$ and $t = 3$. In part (a)(i), the only critical number on the interval $[0, 3]$ was found to be $t = 1$. The corresponding function values at the boundaries and the critical number are as follows:

$$\begin{aligned} s(0) &= -(0^3) + 3 \cdot 0 = 0 \\ s(1) &= -(1^3) + 3 \cdot 1 = 2 \\ s(3) &= -(3^3) + 3 \cdot 3 = -18 \end{aligned}$$

A comparison of the three preceding function values leads to the following absolute maximum and minimum values of $s(t)$ on $[0, 3]$:

Absolute maximum: $\boxed{s(1) = 2}$

Absolute minimum: $\boxed{s(3) = -18}$

(b) i. Determine the total distance traveled by the particle between $t = 0$ and $t = 3$ seconds. Include the correct unit of measurement.

Solution:

The particle moves in the positive direction when $v(t) > 0$ and it moves in the negative direction when $v(t) < 0$.

In part (a), we found that $v(t) = s'(t) = -3t^2 + 3$, and the only value of t on the interval $[0, 3]$ at which $v(t) = s'(t)$ equals zero is $t = 1$. Therefore, the total distance traveled on time interval $[0, 3]$ will be calculated as the sum of the distances traveled during the time intervals $[0, 1]$ and $[1, 3]$.

$$\begin{aligned} D &= |s(1) - s(0)| + |s(3) - s(1)| \\ &= |2 - 0| + |-18 - 2| \\ &= 2 + 20 = \boxed{\text{22 meters}} \end{aligned}$$

ii. Determine the particle's acceleration at $t = 2$ seconds. Include the correct unit of measurement.

Solution:

$$a(t) = v'(t) = \frac{d}{dt}[-3t^2 + 3] = -6t$$

$$a(2) = -6 \cdot 2 = \boxed{-12 \text{ m/s}^2}$$

2. (21 pts) Consider the following relationship: $x \cos y = 3y + 4x^3$.

(a) Find an expression for $\dfrac{dy}{dx}$.

Solution:

Use implicit differentiation.

$$\frac{d}{dx}[x \cos y] = \frac{d}{dx}[3y + 4x^3]$$

$$x \cdot \frac{d}{dx}[\cos y] + \cos y \cdot \frac{d}{dx}[x] = 3 \cdot \frac{dy}{dx} + 12x^2$$

$$x \cdot (-\sin y) \cdot \frac{dy}{dx} + \cos y = 3 \cdot \frac{dy}{dx} + 12x^2$$

$$\frac{dy}{dx} \cdot (-x \sin y - 3) = 12x^2 - \cos y$$

$$\frac{dy}{dx} = \boxed{\frac{\cos y - 12x^2}{x \sin y + 3}}$$

(b) Find an equation of the tangent line to the curve $x \cos y = 3y + 4x^3$ at the point $(1/2, 0)$.

Solution:

We can use the result in part (a) to determine the slope of the tangent line:

$$\left.\frac{dy}{dx}\right|_{(1/2,0)} = \left.\frac{\cos y - 12x^2}{x \sin y + 3}\right|_{x=1/2, y=0}$$

$$= \frac{\cos(0) - 12 \cdot (1/2)^2}{(1/2) \cdot \sin(0) + 3}$$

$$= \frac{1-3}{0+3} = -\frac{2}{3}$$

Therefore, the point-slope form of the tangent line is $\boxed{y = -\frac{2}{3}\left(x - \frac{1}{2}\right)}$

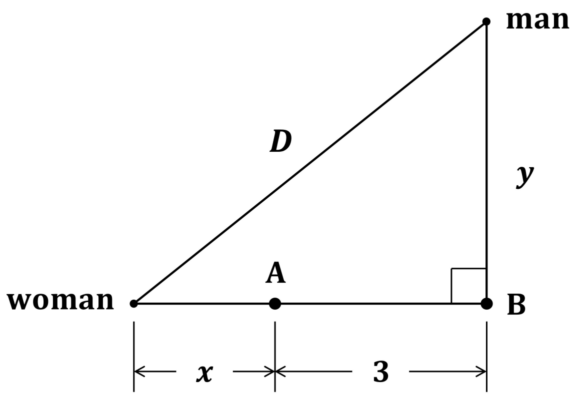
3. (20 pts) Consider two fixed points such that point A is 3 miles west of point B.

Suppose a man starts walking north from point B at a constant speed of 4 miles per hour at the same time a woman starts walking west from point A at a constant rate of 2 miles per hour.

How fast is the distance between the two people increasing when the man is 4 miles north of point B? Include the correct unit of measurement.

Solution:

Let $x = x(t)$ represent the distance between the woman and point A, let $y = y(t)$ represent the distance between the man and point B, and let $D = D(t)$ represent the distance between the two people. The situation can be depicted as follows:



The objective is to determine the value of dD/dt when $y = 4$.

Since the woman is walking at a constant rate of 2 miles per hour, we know that $dx/dt = 2$. Likewise, since the man is walking at a constant rate of 4 miles per hour, we know that $dy/dt = 4$.

The Pythagorean Theorem implies that $(x+3)^2 + y^2 = D^2$. Implementing implicit differentiation by taking the derivative with respect to t of both sides of this equation produces the following:

$$2(x+3) \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 2D \cdot \frac{dD}{dt}$$

$$(x+3) \cdot \frac{dx}{dt} + y \cdot \frac{dy}{dt} = D \cdot \frac{dD}{dt}$$

Let $t = 0$ represent the time at which the two people begin walking from their respective starting points. Since the man is walking at a constant speed of 4 miles per hour, he will be a distance of $y = 4$ miles from his starting point at time $t = 1$ hour.

Since the woman is walking at a constant speed of 2 miles per hour, she will be a distance of $x = 2$ miles from her starting point at that same time ($t = 1$ hour).

At that moment ($t = 1$ hour), the distance between the two people is

$$D = \sqrt{(2+3)^2 + 4^2} = \sqrt{5^2 + 4^2} = \sqrt{41} \text{ miles}$$

Therefore, at time $t = 1$ hour, we have the following:

$$\begin{aligned}
(x+3) \cdot \frac{dx}{dt} + y \cdot \frac{dy}{dt} &= D \cdot \frac{dD}{dt} \\
(2+3) \cdot 2 + 4 \cdot 4 &= \sqrt{41} \cdot \frac{dD}{dt} \\
10 + 16 &= \sqrt{41} \cdot \frac{dD}{dt} \\
\frac{dD}{dt} &= \boxed{\frac{26}{\sqrt{41}} \text{ miles per hour}}
\end{aligned}$$

4. (23 pts) Parts (a) and (b) are unrelated.

(a) i. Find the linearization of $f(x) = \sqrt{1+x}$ centered at $x = 24$.

Solution:

$f(x) \approx L(x) = f(24) + f'(24)(x-24)\ ,\ \ x \approx 24$

$$f(x) = \sqrt{1+x} = (1+x)^{1/2}$$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2} = \frac{1}{2\sqrt{1+x}}$$

$$f(24) = \sqrt{1+24} = \sqrt{25} = 5$$

$$f'(24) = \frac{1}{2\sqrt{1+24}} = \frac{1}{2\sqrt{25}} = \frac{1}{10} = 0.1$$

Therefore, $L(x) = \boxed{5 + 0.1(x-24)}$

ii. Use the linear approximation from part (a) to estimate the value of $\sqrt{26}$.

Solution:

$$\begin{aligned}\sqrt{26} &= \sqrt{1+25} = f(25)\\ &\approx L(25) = 5 + 0.1(25-24)\\ &= 5 + 0.1 \cdot 1 = \boxed{5.1}\end{aligned}$$

(b) The radius of a circle was measured and found to be 20 cm with a possible error in measurement of at most 0.1 cm. Suppose that value of the radius is used to compute the area of the circle. Use differentials to determine the following values. Include the correct unit of measurement where applicable.

i. The maximum possible error in the computed value of the area.

ii. The maximum possible relative error in the computed value of the area.

Solution:

i. The problem statement indicates that $r = 20$ cm and $dr = 0.1$ cm. The error in the computed value of the area is dA.

The area of a circle of radius r is given by $A = \pi r^2$.

$$\frac{dA}{dr} = 2\pi r$$

$$dA = 2\pi r dr$$

$$= 2\pi \cdot 20 \cdot 0.1 = \boxed{4\pi \text{ cm}^2}$$

ii. The relative error in area is given by dA/A.

$$A = \pi r^2 = \pi \cdot 20^2 = 400\pi$$

$$\frac{dA}{A} = \frac{4\pi}{400\pi} = \boxed{0.01}$$

5. (24 pts) Parts (a) and (b) are not related.

(a) Let $g(x) = x^{1/2} - x^{3/2}$ on the interval $[0, 1]$.

i. Verify that $g(x)$ satisfies all three hypotheses of Rolle's Theorem on the specified interval.

Solution:

Since $g(x)$ is a difference of root functions, it is continuous on its entire domain, which includes the closed interval $[0, 1]$. Therefore, $\boxed{g(x) \text{ is continuous on the closed interval } [0, 1]}$

$$g'(x) = \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{1/2}$$

Since $g'(x)$ is defined everywhere on the interval $(0, 1)$, we know that $\boxed{g(x) \text{ is differentiable on the open interval } (0, 1)}$

Finally, we know that $g(0) = 0^{1/2} - 0^{3/2} = 0$ and $g(1) = 1^{1/2} - 1^{3/2} = 0$. Therefore, $\boxed{g(0) = g(1)}$

ii. Find all numbers c that satisfy the conclusion of Rolle's Theorem.

Solution:

Since $g(x)$ satisfies all three hypotheses of Rolle's Theorem on the interval $[0, 1]$, the conclusion of the theorem states that there exists at least one number c on the interval $(0, 1)$ such that $f'(c) = 0$. So, using the expression for $g'(x)$ that was derived in part (a)(i), we have:

$$g'(c) = \frac{1}{2}c^{-1/2} - \frac{3}{2}c^{1/2} = 0$$

$$\frac{1}{2}c^{-1/2}(1 - 3c) = 0$$

$$c = \boxed{1/3}$$

(b) Let $h(x) = x^3 - 3x^2 - x$ on the interval $[1, 4]$.

i. Verify that $h(x)$ satisfies both hypotheses of the Mean Value Theorem on the specified interval.

Solution:

Since $h(x)$ is a polynomial, it is continuous and differentiable on the interval $(-\infty, \infty)$. Therefore,

$\boxed{h(x) \text{ is continuous on the closed interval } [1, 4]}$

$\boxed{h(x) \text{ is differentiable on the open interval } (1, 4)}$

ii. Find all numbers c that satisfy the conclusion of the Mean Value Theorem.

Solution:

Since $h(x)$ satisfies both hypotheses of the Mean Value Theorem on the interval $[1, 4]$, the conclusion of the theorem states that there exists at least one number c on the interval $(1, 4)$ such that:

$$h'(c) = \frac{h(4) - h(1)}{4 - 1}$$

$h'(x) = 3x^2 - 6x - 1$
$h(1) = 1^3 - 3 \cdot 1^2 - 1 = -3$
$h(4) = 4^3 - 3 \cdot 4^2 - 4 = 64 - 48 - 4 = 12$

Therefore, there exists at least one number c on the interval $(1, 4)$ such that:

$$3c^2 - 6c - 1 = \frac{12 - (-3)}{4 - 1}$$

$$3c^2 - 6c - 1 = \frac{15}{3} = 5$$

$$3c^2 - 6c - 6 = 0$$

$$c^2 - 2c - 2 = 0$$

$$c = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-2)}}{2 \cdot 1}$$

$$c = \frac{2 \pm \sqrt{12}}{2} = 1 \pm \sqrt{3}$$

Of those two values of c, the only one that is on the interval $(1, 4)$ is $c = \boxed{1 + \sqrt{3}}$

6. (15 pts) Consider the rational function $r(x) = \dfrac{x^2+2x+1}{1-x^2}$.

   (a) Find all values of x corresponding to removable discontinuities of $r(x)$, if any exist. If no removable discontinuities exist, clearly state "none". Support your answer by evaluating the appropriate limit(s).

   (b) Find the equation of every vertical asymptote of $y = r(x)$, if any exist. If no vertical asymptotes exist, clearly state "none". Support your answer by evaluating the appropriate limit(s).

Solution:

$$r(x) = \frac{x^2+2x+1}{1-x^2} = \frac{(x+1)^2}{(1-x)(1+x)} = \frac{x+1}{1-x}, \quad x \neq \pm 1$$

(a) Since direct substitution into the preceding simplified expression for $r(x)$ does not involve division by zero for $x = -1$, a two-sided limit can be evaluated as x approaches -1.

$$\lim_{x\to -1} r(x) = \lim_{x\to -1} \frac{x^2+2x+1}{1-x^2} = \lim_{x\to -1} \frac{x+1}{1-x} = \frac{-1+1}{1-(-1)} = \frac{0}{2} = 0$$

Since the preceding limit is finite, $r(x)$ has a removable discontinuity at $\boxed{x = -1}$

(b) Since direct substitution into the simplified expression for $r(x)$ involves division by zero for $x = 1$, a one-sided limit must be evaluated as x approaches 1.

$$\lim_{x\to 1^-} r(x) = \lim_{x\to 1^-} \frac{x^2+2x+1}{1-x^2} = \lim_{x\to 1^-} \frac{x+1}{1-x} \to \frac{2}{0^+} = \infty$$

$$\lim_{x\to 1^+} r(x) = \lim_{x\to 1^+} \frac{x^2+2x+1}{1-x^2} = \lim_{x\to 1^+} \frac{x+1}{1-x} \to \frac{2}{0^-} = -\infty$$

Since at least one of the preceding two limits is infinite (in fact, both are infinite for this function), $r(x)$ has a vertical asymptote at $\boxed{x = 1}$

7. (21 pts) Parts (a) and (b) are unrelated.

(a) Determine $u'(x)$ for the function $u(x) = 5x - x^2$ by using the definition of derivative. You must obtain u' by evaluating an appropriate limit to earn credit.

Solution:

The definition of derivative indicates that $u'(x) = \lim\limits_{h\to 0} \dfrac{u(x+h) - u(x)}{h}$.

$$\begin{aligned} u'(x) &= \lim_{h\to 0} \frac{\big(5(x+h) - (x+h)^2\big) - (5x - x^2)}{h} \\ &= \lim_{h\to 0} \frac{5x + 5h - (x^2 + 2xh + h^2) - 5x + x^2}{h} \\ &= \lim_{h\to 0} \frac{5h - 2xh - h^2}{h} \\ &= \lim_{h\to 0} 5 - 2x - h = \boxed{5 - 2x} \end{aligned}$$

(b) Consider the function $w(x)$, which is defined as follows:

$$w(x) = \begin{cases} \dfrac{\sin(3x)}{x} & , \quad x < 0 \\ \dfrac{x^2 - 2x - 8}{x - 4} & , \quad x > 0 \end{cases}$$

i. Evaluate both one-sided limits of $w(x)$ as x approaches zero.

ii. Briefly explain why $w(x)$ has a jump discontinuity at $x = 0$.

Solution:

i.

$$\lim_{x \to 0^-} w(x) = \lim_{x \to 0^-} \frac{\sin(3x)}{x} = \lim_{x \to 0^-} \frac{3\sin(3x)}{3x} = 3 \lim_{x \to 0^-} \frac{\sin(3x)}{3x} = 3 \cdot 1 = \boxed{3}$$

$$\lim_{x \to 0^+} w(x) = \lim_{x \to 0^+} \frac{x^2 - 2x - 8}{x - 4} = \lim_{x \to 0^+} \frac{(x+2)(x-4)}{x - 4} = \lim_{x \to 0^+} (x + 2) = \boxed{2}$$

ii. $w(x)$ has a jump discontinuity at $x = 0$ because $\boxed{\lim_{x \to 0^-} w(x) \neq \lim_{x \to 0^+} w(x)}$