Potentially useful formulas:

1. $a^3 - b^3 = (a-b)(a^2+ab+b^2)$
2. Circle: $(x-h)^2 + (y-k)^2 = r^2$
3. Arc length: $s = r\theta$
4. $\sin(a-b) = \sin a \cos b - \sin b \cos a$
5. $\cos(a-b) = \cos a \cos b + \sin a \sin b$
6. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
7. $\cos(2\theta) = 2\cos^2\theta - 1$
8. $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$
9. $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$
10. $a^3 + b^3 = (a+b)(a^2-ab+b^2)$
11. Area of a sector: $A = \frac{1}{2}r^2\theta$
12. $\sin(a+b) = \sin a \cos b + \sin b \cos a$
13. $\cos(a+b) = \cos a \cos b - \sin a \sin b$
14. $\sin(2\theta) = 2\sin\theta\cos\theta$
15. $\cos(2\theta) = 1 - 2\sin^2\theta$
16. $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$
17. $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$

---

1. The following are unrelated. (7 pts)

(a) Perform the indicated operations: $\frac{5}{18} + \frac{1}{12} - 2^{-1}$

Solution:

$$\begin{aligned}
\frac{5}{18} + \frac{1}{12} - 2^{-1} &= \frac{5}{18} + \frac{1}{12} - \frac{1}{2} && (1)\\
&= \frac{10}{36} + \frac{3}{36} - \frac{18}{36} && (2)\\
&= \frac{(10+3-18)}{36} && (3)\\
&= \boxed{-\frac{5}{36}} && (4)
\end{aligned}$$

(b) Evaluate the expression: $-\frac{\sqrt{18}}{2\sqrt{32}}$

Solution:

$$\begin{aligned}
-\frac{\sqrt{18}}{2\sqrt{32}} &= -\frac{\sqrt{2\cdot 9}}{2\sqrt{2\cdot 16}} && (5)\\
&= -\frac{3\sqrt{2}}{2\cdot 4\sqrt{2}} && (6)\\
&= \boxed{-\frac{3}{8}} && (7)
\end{aligned}$$

2. Rewrite each of the following without absolute value symbol (4 pts):

(a) $|z+5|$ where $z < -5$

Solution:

Since $z < -5$ we know $z+5 < 0$. Hence $|z+5| = \boxed{-(z+5)}$ or $\boxed{-z-5}$

(b) $|e-1|$

Solution:

Since $e > 1$ we know $e-1 > 0$. Hence $|e-1| = \boxed{e-1}$

3. Let x and y be real numbers such that $x \geq 0$ and $y < 0$. Determine whether the following expression is positive, negative, or the sign cannot be determined (2 pts):

$$-x^2y^3 - 2y$$

Solution:

Since $y < 0$, the term $-2y > 0$.

If $x = 0$ then the quantity $-x^2y^3 - 2y$ simplifies to $-2y$ which is positive.

If $x > 0$, then we know that $x^2 > 0$ for any x. Since $y < 0$, we know $y^3 < 0$. Hence the term $x^2y^3 < 0$ and the term $-x^2y^3 > 0$.

Thus, in the case $x > 0$, the quantity $-x^2y^3 - 2y$ is a sum of two positive numbers, and is thus positive.

So the answer in both cases is $\boxed{\text{positive}}$.

4. The following are unrelated. (15 pts)

(a) Perform the indicated operations: $5x(2x - x^3) + (2x^2 - 3)^2$

Solution:

$$\begin{aligned} 5x(2x - x^3) + (2x^2 - 3)^2 &= 10x^2 - 5x^4 + \left(2x^2\right)^2 - 2(2x^2)3 + 3^2 && (8) \\ &= 10x^2 - 5x^4 + 4x^4 - 12x^2 + 9 && (9) \\ &= \boxed{-x^4 - 2x^2 + 9} && (10) \end{aligned}$$

(b) Perform the indicated operations (leave answer without negative exponents):

$$(-xy^4)^2 \left(\frac{xy^{-3}}{2}\right) + 2\frac{x^{\frac{9}{2}}}{\sqrt{x}}$$

Solution:

$$\begin{aligned} (-xy^4)^2 \left(\frac{xy^{-3}}{2}\right) + 2\frac{x^{\frac{9}{2}}}{\sqrt{x}} &= x^2y^8 \left(\frac{xy^{-3}}{2}\right) + 2\frac{x^{\frac{9}{2}}}{x^{\frac{1}{2}}} && (11) \\ &= \frac{x^{(2+1)}y^{(8-3)}}{2} + 2x^{\left(\frac{9}{2}-\frac{1}{2}\right)} && (12) \\ &= \boxed{\frac{x^3y^5}{2} + 2x^4} && (13) \end{aligned}$$

(c) Perform the indicated operation: $\dfrac{x}{2x-26} - \dfrac{1}{x^2-13x}$

Solution:

$$\frac{x}{2x-26} - \frac{1}{x^2-13x} = \frac{x}{2(x-13)} - \frac{1}{x(x-13)} \tag{14}$$

$$= \frac{x^2}{2x(x-13)} - \frac{2}{2x(x-13)} \tag{15}$$

$$= \boxed{\frac{x^2-2}{2x(x-13)}} \tag{16}$$

5. The following are unrelated. (9 pts)

(a) Perform the indicated operations: $\left(2^{-x} + 3 \cdot 2^{3x}\right) 2^{3x} + (2^x)^6$

Solution:

$$\left(2^{-x} + 3 \cdot 2^{3x}\right) 2^{3x} + (2^x)^6 = 2^{-x} \cdot 2^{3x} + 3 \cdot 2^{3x} \cdot 2^{3x} + 2^{6x} \tag{17}$$

$$= 2^{2x} + 3 \cdot 2^{3x+3x} + 2^{6x} \tag{18}$$

$$= 2^{2x} + 3 \cdot 2^{6x} + 2^{6x} \tag{19}$$

$$= \boxed{2^{2x} + 4 \cdot 2^{6x}} \tag{20}$$

(b) Evaluate the following: $10^{2\log(3)} + \log_3\left(3^{x-1}\right) - \ln(1)$ (Your answer should have no logarithms)

Solution:

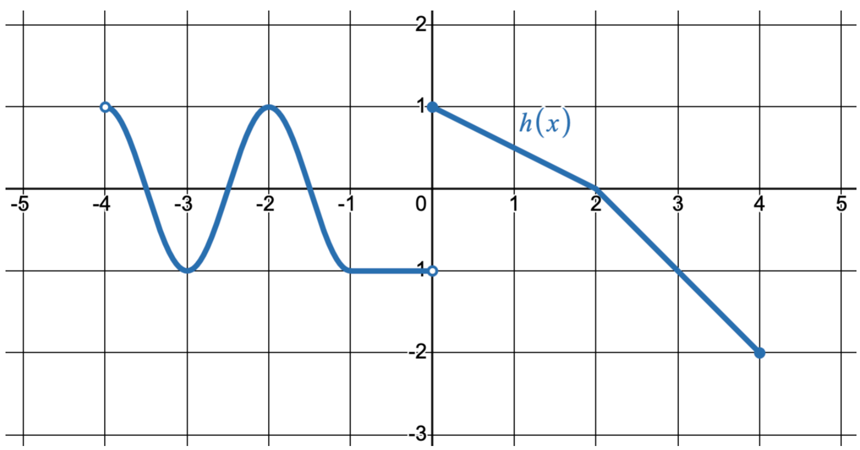
$$10^{2\log(3)} + \log_3\left(3^{x-1}\right) - \ln(1) = 10^{\log(3^2)} + x - 1 - 0 \tag{21}$$

$$= 3^2 + x - 1 \tag{22}$$

$$= 9 + x - 1 \tag{23}$$

$$= \boxed{8 + x} \tag{24}$$

6. Use the graph of $h(x)$ below to answer the following (11 pts):



(a) Identify the domain of $h(x)$.

Solution: $\boxed{\text{Domain: } (-4, 4]}$

(b) Identify the range of $h(x)$.

Solution: $\boxed{\text{Range: } [-2, 1]}$

(c) Solve the equation $h(x) = 1$.

Solution: $h(0) = 1$ and $h(-2) = 1$. The solutions of $h(x) = 1$ are $\boxed{x = -2, 0}$

(d) If $g(x) = x^2 - \sin(x + 1)$ find $g(h(3))$

Solution:

$$\begin{aligned} g(h(3)) &= g(-1) && (25)\\ &= (-1)^2 - \sin(-1 + 1) && (26)\\ &= 1 - \sin(0) && (27)\\ &= 1 - 0 && (28)\\ &= \boxed{1} && (29) \end{aligned}$$

(e) $h(x)$ is not currently a one-to-one function. Identify a restriction of the domain that preserves the range and results in a one-to-one function. Give answer in interval notation.

One possible solution (there are other possibilities): $\boxed{\text{Domain:}[0, 4]}$

7. Solve the following equations for the indicated variable. If there are no solutions, write **no solutions**. (12 pts)

(a) Solve for x: $4 - x^2 = 5x$

Solution:

$$\begin{aligned} 4 - x^2 &= 5x && (30)\\ 0 &= x^2 + 5x - 4 && (31) \end{aligned}$$

Using quadratic formula

$$\begin{aligned} x &= \frac{-5 \pm \sqrt{5^2 - 4 \cdot 1 \cdot (-4)}}{2} && (32)\\ &= \frac{-5 \pm \sqrt{25 + 16}}{2} && (33)\\ &= \boxed{\frac{-5 \pm \sqrt{41}}{2}} && (34) \end{aligned}$$

(b) Solve for k: $k = \frac{1}{2}n(1 - k)$

Solution:

$$\begin{aligned} k &= \frac{1}{2}n(1 - k) && (35)\\ 2(k) &= 2\left(\frac{1}{2}n(1 - k)\right) && (36)\\ 2k &= n(1 - k) && (37)\\ 2k &= n - nk && (38)\\ 2k + nk &= n && (39)\\ k(2 + n) &= n && (40)\\ k &= \boxed{\frac{n}{2 + n}} && (41) \end{aligned}$$

(c) Solve for r: $3e^{r-1} + 4 = 2$

Solution:

$$3e^{r-1} + 4 = 2 \tag{42}$$
$$3e^{r-1} = -2 \tag{43}$$
$$e^{r-1} = -\frac{2}{3} \tag{44}$$

As the exponential function $y = e^{r-1}$ has range $(0, \infty)$, there are no solutions to the equation $e^{r-1} = -\frac{2}{3}$. $\boxed{\text{No solutions}}$

8. A chemist measures the volume, V, of a certain substance and also its temperature, t. Volume has units of centimeters cubed and temperature has units of degrees Celsius. The volume of the substance is 8 centimeters cubed when its temperature is 5 degrees Celsius and the volume is 11 centimeters cubed when the temperature is 7 degrees Celsius. The chemist finds that the volume of the substance is a linear function of its temperature: $V(t) = mt + b$. Use this information to answer the following. (11 pts)

(a) What is the value of the slope of the linear function $V(t) = mt + b$?

Solution: We have $V(5) = 8$ and $V(7) = 11$. Using the slope formula $m = \dfrac{V_2 - V_1}{t_2 - t_1}$,

$$m = \frac{11 - 8}{7 - 5} = \frac{3}{2} \tag{45}$$

$\boxed{m = \frac{3}{2}}$

(b) What are the units of the slope?

Solution: The units of the slope are the units of V divided by the units of t: Centimeters cubed divided by degrees Celsius or $\boxed{\text{cm}^3/°\text{C}}$.

(c) Find the equation of the linear function, $V(t) = mt + b$, for this particular substance.

Solution: First, we need to find b. Use the information that $V(5) = 8$ and $m = \frac{3}{2}$ to find b.

$$V(5) = \frac{3}{2} \cdot 5 + b \tag{46}$$
$$8 = \frac{15}{2} + b \tag{47}$$
$$b = 8 - \frac{15}{2} \tag{48}$$
$$b = 8 \cdot \frac{2}{2} - \frac{15}{2} \tag{49}$$
$$b = \frac{16 - 15}{2} \tag{50}$$
$$b = \frac{1}{2} \tag{51}$$

The equation of the linear function for this particular substance is $\boxed{V(t) = \frac{3}{2}t + \frac{1}{2}}$.

(d) Use your answer from part (c) to find the temperature of the substance when its volume is 16 centimeters cubed.

Solution: Find t when $V(t) = 16$ cm^3.

$$\begin{aligned} V(t) &= 16 && (52)\\ \frac{3}{2}t + \frac{1}{2} &= 16 && (53)\\ 2\left(\frac{3}{2}t + \frac{1}{2}\right) &= 2(16) && (54)\\ 3t + 1 &= 32 && (55)\\ 3t &= 31 && (56)\\ t &= \frac{31}{3} && (57) \end{aligned}$$

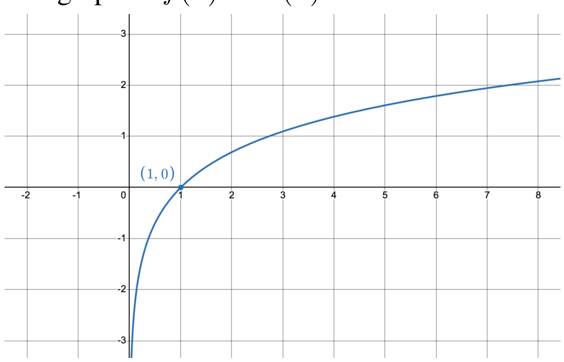
When $\boxed{t = \frac{31}{3}\ ^\circ\text{C}}$, $V(t) = 16$ cm^3.

9. Consider the functions: $f(x) = \ln(x)$ and $g(x) = \dfrac{1}{x-2}$. (9 pts)

(a) Fill in the blank for the function $f(x)$: $f(x) \to$ ____ as $x \to \infty$.

Solution:

The graph of $f(x) = \ln(x)$ is:



From the graph we can see that $f(x) \to \boxed{\infty}$ as $x \to \infty$

(b) Find $(g \circ f)(x)$.

Solution:

$$\begin{aligned} (g \circ f)(x) &= (g(f(x))) && (58)\\ &= g(\ln(x)) && (59)\\ &= \boxed{\frac{1}{\ln(x) - 2}} && (60) \end{aligned}$$

(c) Find the domain of $(g \circ f)(x)$. Give your answer in interval notation.

Solution:

Considering our solution from part (b) we need to consider the domain of $\ln(x)$ and make sure that $\ln(x) - 2 \neq 0$. $\ln(x)$ has domain $(0, \infty)$. Solving $\ln(x) - 2 = 0$ we get:

$$\begin{aligned} \ln(x) - 2 &= 0 && (61) \\ \ln(x) &= 2 && (62) \\ x &= e^2 && (63) \end{aligned}$$

So the domain of $(g \circ f)(x)$ is $\boxed{(0, e^2) \cup (e^2, \infty)}$.

10. For the rational function $R(x) = \dfrac{x^2 - 9}{x^3 + 12x^2 + 27x}$ answer the following: (10 pts)

(a) Find the x-coordinate of any hole(s). If there are no hole(s) write NONE.

Solution:

$R(x) = \dfrac{x^2 - 9}{x^3 + 12x^2 + 27x}$ has a hole when factors between numerator and denominator cancel. So we begin by factoring numerator and denominator:

$$\begin{aligned} R(x) &= \frac{x^2 - 9}{x^3 + 12x^2 + 27x} && (64) \\ &= \frac{(x-3)(x+3)}{x\left(x^2 + 12x + 27\right)} && (65) \\ &= \frac{(x-3)(x+3)}{x(x+9)(x+3)} && (66) \end{aligned}$$

Terms do cancel, specifically $x + 3$ so $\dfrac{(x-3)(x+3)}{x(x+9)(x+3)} = \dfrac{x-3}{x(x+9)}$. The x-coordinate of a hole is the x-value that makes $x + 3 = 0$. $\boxed{x = -3}$

(b) Find the y-coordinate of any hole(s). If there are no hole(s) write NONE.

Solution:

Since the only difference between the graph of $R(x)$ and $y = \dfrac{x-3}{x(x+9)}$ is that $R(x)$ has a hole at $x = -3$ and $y = \dfrac{x-3}{x(x+9)}$ does not, we can use the simplified function to find the y-coordinate of the hole.

$y = \dfrac{-3-3}{-3(-3+9)} = \dfrac{-6}{-18} = \boxed{\dfrac{1}{3}}$

(c) Identify the horizontal or slant asymptote of $R(x)$. If there is no horizontal or slant asymptote write NONE.

Solution:

Since the degree of the polynomial in the numerator is 2 and the denominator is 3 we can take the ratio of the leading terms to determine if there is a horizontal asymptote or not.

As $x \to \pm\infty$ then $R(x) = \dfrac{x^2 - 9}{x^3 + 12x^2 + 27x} \approx \dfrac{x^2}{x^3} = \dfrac{1}{x} \to 0.$

So there is a horizontal asymptote of $\boxed{y = 0}$.

(d) Find all vertical asymptote(s). If there are none write NONE.

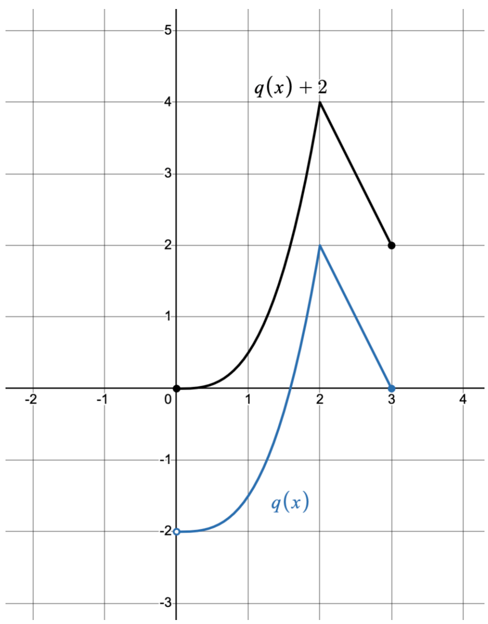
Solution:

We use the simplified function $y = \dfrac{x-3}{x(x+9)}$ to determine the vertical asymptotes. Since no other factors cancel then there are vertical asymptotes when $x(x+9) = 0$. The vertical asymptotes are: $\boxed{x = 0, x = -9}$.

11. For the graph of the function $q(x)$ below use transformations to sketch the graph of $q(x) + 2$ **on the same set of axes**. (3 pts)

Solution:

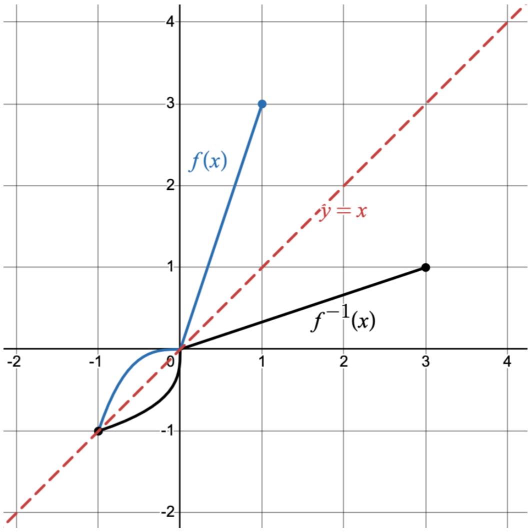
$q(x) + 2$ is the graph of $q(x)$ shifted up 2.



12. For the graph of $f(x)$ below, sketch the graph of $f^{-1}(x)$ **on the same set of axes below**. (4 pts)

Solution:

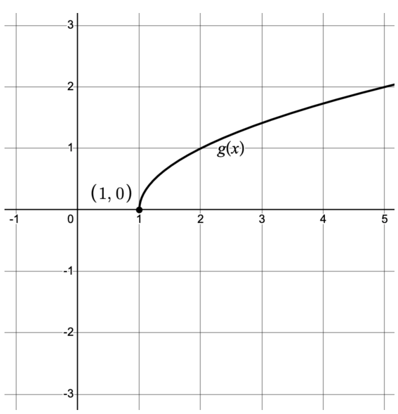
The graph of $f^{-1}(x)$ is found by taking the graph of $f(x)$ and reflecting it across the line $y = x$. We can find points to help us graph $f^{-1}(x)$ by swapping the x and y coordinates of points on $f(x)$. Specifically the points $(-1, -1)$ and $(1, 3)$ on the graph of $f(x)$ correspond to points $(-1, -1)$ and $(3, 1)$ on the graph of $f^{-1}(x)$.



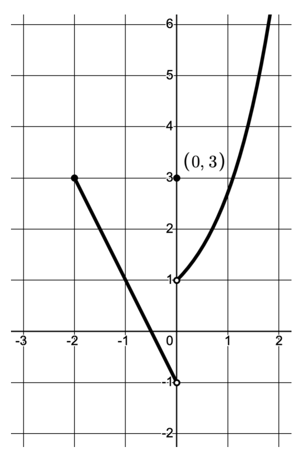
13. Sketch the graph of the following functions. **Label all intercepts and asymptotes as appropriate.** (14 pts)

(a) $g(x) = \sqrt{x-1}$

Solution:



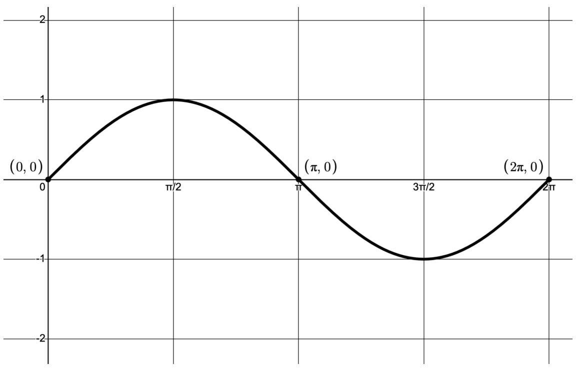
(b) $n(x) = \begin{cases} -2x-1 & \text{if} \quad -2 \leq x < 0 \\ 3 & \text{if} \quad x = 0 \\ e^x & \text{if} \quad x > 0 \end{cases}$

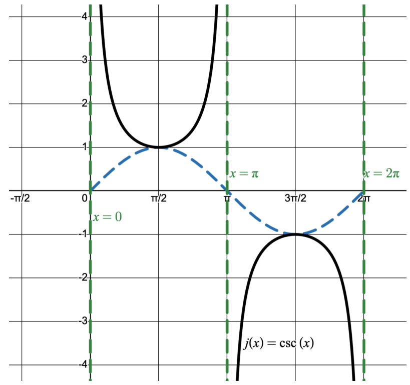


(c) $r(x) = \sin(x)$ on the restricted domain $[0, 2\pi]$

(d) One cycle of $j(x) = \csc(x)$

Solution:





14. Find the exact value for each (do not attempt to find decimal approximations): (13 pts)

(a) $\cos\left(\frac{5\pi}{6}\right)$

(b) $\sin^{-1}\left(\frac{1}{2}\right)$

Solution:

$$\cos\left(\frac{5\pi}{6}\right) = \boxed{-\frac{\sqrt{3}}{2}}$$

$$\sin^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{6}}$$

(c) $\tan(225°)$

(d) $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$

Solution:

$$\tan(225°) = \boxed{1}$$

$$\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right) = \cos^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{3}}$$

15. Find all solutions to the following equations: (8 pts)

(a) $2\cos(\theta) - \sqrt{2} = 0$

Solution: We first solve for $\cos(\theta)$:

$$2\cos(\theta) - \sqrt{2} = 0 \tag{67}$$
$$2\cos(\theta) = \sqrt{2} \tag{68}$$
$$\cos(\theta) = \frac{\sqrt{2}}{2} \tag{69}$$

The two angles on $[0, 2\pi)$ that solve our equation are $\frac{\pi}{4}$ and $\frac{7\pi}{4}$. To get all of the solutions, we add an integer multiple of 2π to each:

$$\theta = \boxed{\frac{\pi}{4} + 2k\pi} \tag{70}$$
$$\text{and} \tag{71}$$
$$\theta = \boxed{\frac{7\pi}{4} + 2k\pi} \tag{72}$$

where k is any integer.

(b) $\tan\theta\sin\theta - \tan\theta = 0$

Solution: We first need to factor:

$$\tan\theta\sin\theta - \tan\theta = 0 \quad (73)$$
$$\tan\theta(\sin\theta - 1) = 0 \quad (74)$$

We now use the multiplicative property of 0 to determine that either

$$\tan\theta = 0 \quad (75)$$
$$\text{or} \quad (76)$$
$$\sin\theta = 0 \quad (77)$$
$$(78)$$

Since $\tan\theta = 0$ and the period of tangent is π, the first equation gives solutions

$$\theta = \boxed{k\pi} \quad (79)$$

for integer values of k. Since $\sin\frac{\pi}{2} = 0$, and $\sin\theta$ has period 2π, the second equation gives

$$\theta = \boxed{\frac{\pi}{2} + 2k\pi} \quad (80)$$

again for integer k.

16. For $m(x) = 5\cos\left(\frac{x}{2}\right)$ (7 pts)

(a) Identify the amplitude.

Solution:

$$\boxed{5}$$

(b) Identify the period.

Solution:

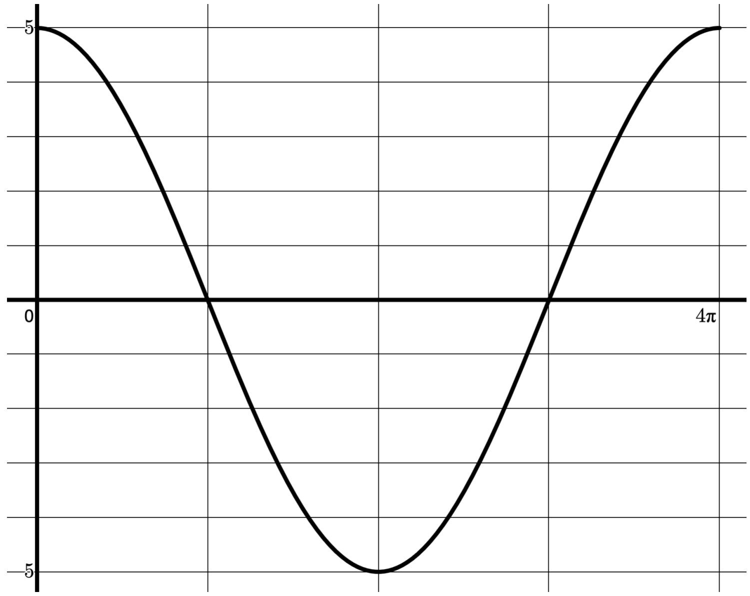
$$P = \frac{2\pi}{\frac{1}{2}} \quad (81)$$
$$= \boxed{4\pi} \quad (82)$$

(c) Identify the phase shift.

Solution:

$$\boxed{0}$$

(d) Sketch one cycle of the graph of $m(x)$. **Label** at least two values on the x-axis and clearly identify the amplitude.



17. Verify the identity: $\cos\alpha\cot\alpha = \csc\alpha - \sin\alpha$. (5 pts)

Solution:

Here are two possible solutions (there are others):

If we start with the left hand side, and utilize $\cos^2\alpha = 1 - \sin^2\alpha$, then we have:

$$\cos\alpha\cot\alpha = \cos\alpha\frac{\cos\alpha}{\sin\alpha} \tag{83}$$
$$= \frac{\cos^2\alpha}{\sin\alpha} \tag{84}$$
$$= \frac{1-\sin^2\alpha}{\sin\alpha} \tag{85}$$
$$= \frac{1}{\sin\alpha} - \frac{\sin^2\alpha}{\sin\alpha} \tag{86}$$
$$= \csc\alpha - \sin\alpha \tag{87}$$

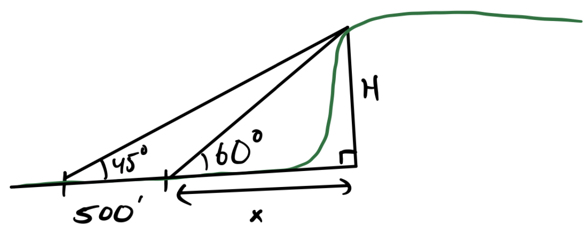
If we start on the right hand side, and utilize $1 - \sin^2\alpha = \cos^2\alpha$, then we have:

$$\csc\alpha - \sin\alpha = \frac{1}{\sin\alpha} - \sin\alpha \tag{88}$$
$$= \frac{1}{\sin\alpha} - \frac{\sin^2\alpha}{\sin\alpha} \tag{89}$$
$$= \frac{1-\sin^2\alpha}{\sin\alpha} \tag{90}$$
$$= \frac{\cos^2\alpha}{\sin\alpha} \tag{91}$$
$$= \cos\alpha\frac{\cos\alpha}{\sin\alpha} \tag{92}$$
$$= \cos\alpha\cot\alpha \tag{93}$$

Either way, we have verified the identity.

18. To estimate the height of a cliff above a level plain, a surveyor measures the angle of elevation to the top of the cliff to be 45°. Five hundred feet closer to the cliff along the plain, the angle of elevation is found to be 60°. Find the exact height of the cliff (do not attempt to round to a decimal number). (6 pts)

Solution: We first sketch the terrain and draw the right triangles on it:



We see we have two unknowns: the height of the cliff, H, and the distance from the cliff, x. Since we know the angles, we can use their tangents to create the equations involving H and x:

$$\tan 45^\circ = \frac{H}{x+500} \tag{94}$$

$$\tan 60^\circ = \frac{H}{x} \tag{95}$$

Since we are trying to find H, let's solve the second equation for x, substitute it into the first equation, then solve for H. Evaluating the tangent and solving for x gives

$$\tan 60^\circ = \frac{H}{x} \tag{96}$$

$$\sqrt{3} = \frac{H}{x} \tag{97}$$

$$x\sqrt{3} = H \tag{98}$$

$$x = \frac{H}{\sqrt{3}} \tag{99}$$

Now substitue x into the second equation, evaluate the tangent, and solve for H:

$$\tan 45^\circ = \frac{H}{x+500} \tag{100}$$

$$1 = \frac{H}{\frac{H}{\sqrt{3}}+500} \tag{101}$$

$$\frac{H}{\sqrt{3}} + 500 = H \tag{102}$$

$$\sqrt{3}\left(\frac{H}{\sqrt{3}}+500\right) = \sqrt{3}\,(H) \tag{103}$$

$$H + 500\sqrt{3} = \sqrt{3}H \tag{104}$$

$$500\sqrt{3} = \sqrt{3}H - H \tag{105}$$

$$500\sqrt{3} = H\left(\sqrt{3}-1\right) \tag{106}$$

$$H = \boxed{\frac{500\sqrt{3}}{\sqrt{3}-1}\text{ feet}} \tag{107}$$