1. Do each of the following converge or diverge? If it converges, find its limit. If it diverges, explain.

   (a) (8 points) $\displaystyle\sum_{n=2}^{\infty} \frac{n+4}{(n^4-2)^{1/3}}$

   (b) (18 points) $\displaystyle\int_0^{\infty} \frac{3}{(x+1)(x^2+2)}\, dx$

Solution:

(a) Show the series diverges by Direct Comparison:

$$\frac{n+4}{(n^4-2)^{1/3}} \geq \frac{n+4}{n^{4/3}} \geq \frac{n}{n^{4/3}} \geq \frac{1}{n^{1/3}} > 0$$

Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1/3}}$ diverges (p-series) with $p = 1/3 < 1$, the original series diverges by the Direct Comparison Test.

Alternate Solution: The Limit Comparison Test can also be used. Observe that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1/3}}$ diverges (p-series) with $p = 1/3 < 1$. Then, set $a_n = \dfrac{n+4}{(n^4-2)^{1/3}}$ and $b_n = 1/n^{1/3}$ and compute

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+4}{(n^4-2)^{1/3}}}{\frac{1}{n^{1/3}}} = \lim_{n\to\infty} \frac{(n+4)n^{1/3}}{(n^4-2)^{1/3}} = \lim_{n\to\infty} \frac{n^{4/3}+4n^{1/3}}{(n^4-2)^{1/3}} \cdot \frac{1/n^{4/3}}{1/n^{4/3}} = \lim_{n\to\infty} \frac{1+4/n}{(1-2/n^4)^{1/3}} = 1$$

Thus, the Limit Comparison Test, the original series diverges.

(b) This is an improper integral. We first need to use partial fractions on the integrand.

$$\frac{3}{(x+1)(x^2+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2}$$

This implies

$$3 = A(x^2+2) + (Bx+C)(x+1)$$

We can solve for A, B, and C in several ways. One way is to set $x = -1$ to obtain $3 = 3A$ which implies $A = 1$. Set $x = 0$ to obtain $3 = 2A + C$, which implies $C = 1$. Finally, set $x = 1$ and obtain $B = -1$. Thus,

$$\begin{aligned}
\int_0^{\infty} \frac{3}{(x+1)(x^2+2)} &= \int_0^{\infty} \left(\frac{1}{x+1} + \frac{-x+1}{x^2+2}\right) \\
&= \lim_{t\to\infty} \int_0^t \left(\frac{1}{x+1} - \frac{x}{x^2+2} + \frac{1}{x^2+2}\right)\, dx \\
&= \lim_{t\to\infty} \left(\ln|x+1| - (1/2)\ln|x^2+2| + (1/\sqrt{2})\arctan(x/\sqrt{2})\right)\Big|_0^t \\
&= \lim_{t\to\infty} \left(\ln|t+1| - (1/2)\ln|t^2+2| + (1/\sqrt{2})\arctan(t/\sqrt{2}) - (1/2)\ln(2)\right) \\
&= \lim_{t\to\infty} \left(\ln\left|\frac{t+1}{(t^2+2)^{1/2}}\right| + (1/\sqrt{2})\arctan(t/\sqrt{2}) + (1/2)\ln(2)\right) \\
&= \boxed{\frac{\pi}{2\sqrt{2}} + \frac{1}{2}\ln(2)}
\end{aligned}$$

where we used $\displaystyle\lim_{t\to\infty} \ln\left|\frac{t+1}{(t^2+2)^{1/2}}\right| = \ln(1) = 0$ and $\displaystyle\lim_{t\to\infty} \arctan(t/\sqrt{2}) = \frac{\pi}{2}$.

2. Two unrelated problems.

   (a) (12 points) Let $S = \sum_{n=1}^{\infty} a_n$ and suppose the n^{th} partial sum is $s_n = \frac{5}{2}\left(1 - \frac{1}{5^n}\right)$.

      i. Find an expression for a_1, a_2 and a_n.

      ii. Does $S = \sum_{n=1}^{\infty} a_n$ converge? If so, find its limit. If not, explain.

   (b) (18 points) Suppose $g(x) = \dfrac{\ln(1+x) - x}{x^2}$

      i. Find the Maclaurin series for $g(x)$ (use sigma notation). Use $g(0) = -1/2$.

      ii. Find the tenth derivative of g at 0, that is, find $g^{(10)}(0)$.

Solution:

(a) i.

$$\begin{aligned}
a_1 &= s_1 = \frac{5}{2}\left(1 - \frac{1}{5}\right) = \frac{5}{2} \cdot \frac{4}{5} = 2 \\
a_2 &= s_2 - s_1 = \frac{5}{2}\left(1 - \frac{1}{5^2}\right) - \frac{5}{2}\left(1 - \frac{1}{5}\right) = \frac{2}{5} \\
a_n &= s_n - s_{n-1} = \frac{5}{2}\left(1 - \frac{1}{5^n}\right) - \frac{5}{2}\left(1 - \frac{1}{5^{n-1}}\right) = \frac{5}{2}\left(\frac{1}{5^{n-1}} - \frac{1}{5^n}\right) = \frac{2}{5^{n-1}}
\end{aligned}$$

ii. Yes, the series does converge, because its partial sums converge:

$$S = \sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{5}{2}\left(1 - \frac{1}{5^n}\right) = \frac{5}{2}.$$

(b) i.

$$\begin{aligned}
g(x) &= \frac{\ln(1+x) - x}{x^2} \\
&= \frac{\left(\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}\right) - x}{x^2} \\
&= \frac{\sum_{n=2}^{\infty}(-1)^{n-1}\frac{x^n}{n}}{x^2} \\
&= \sum_{n=2}^{\infty}(-1)^{n-1}\frac{x^{n-2}}{n}
\end{aligned}$$

ii. In this series formula, the x has an exponent of 10 when $n = 12$. Hence, the general formula for Taylor series says

$$\frac{g^{(10)}(0)}{10!} = \frac{(-1)^{12-1}}{12} = -\frac{1}{12}.$$

Therefore,

$$g^{(10)}(0) = -\frac{10!}{12}.$$

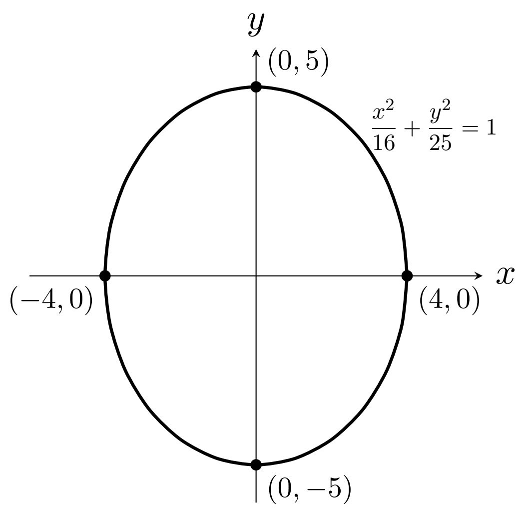
3. (16 points) Consider the equation $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$.

   (a) Graph this equation. Label all intercepts.

(b) Rotate the area bounded by this equation around the line $x = 6$. Set up, but do not evaluate, an integral to find the volume of this solid.

Solution:

(a) The equation $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$ is an ellipse with vertices at $(\pm 4, 0)$ and $(0, \pm 5)$. Visually, we have



(b) **Cylindrical shells:** Solving for y, we find

$$y = \pm\sqrt{25\left(1 - \frac{x^2}{16}\right)}$$

where the postive is the top half of the ellipse and the negative is the bottom half. The height of our region is (top) - (bottom) which gives $h = 2\sqrt{25\left(1 - \frac{x^2}{16}\right)}$. The radius is given by $r = 6 - x$. Putting everything together, the volume can be computed as

$$\boxed{V = \int_{-4}^{4} 2\pi(6 - x)2\sqrt{25\left(1 - \frac{x^2}{16}\right)}\, dx.}$$

Disk/washer: Solving for x, we find

$$x = \pm\sqrt{16\left(1 - \frac{y^2}{25}\right)}.$$

The negative corresponds to the left half of the ellipse while the positive corresponds to the right half. The two radii for the washer method are

$$r = 6 - \sqrt{16\left(1 - \frac{y^2}{25}\right)} \qquad R = 6 + \sqrt{16\left(1 - \frac{y^2}{25}\right)}.$$

Then, the washer method yields the volume

$$\boxed{V = \pi\int_{-5}^{5} \left(6 + \sqrt{16\left(1 - \frac{y^2}{25}\right)}\right)^2 - \left(6 - \sqrt{16\left(1 - \frac{y^2}{25}\right)}\right)^2\, dy.}$$

4. (28 points) Let $f(x) = e^{-3x}$.

   (a) Find the Taylor series (use sigma notation) for f **centered at** $a = 2$.

   (b) Find the radius of convergence of the series. Justify your answer using appropriate test(s).

   (c) Find $T_2(x)$, the 2nd order Taylor polynomial.

   (d) Use Taylor's formula to find an error bound if $T_2(x)$ is used to approximate $f(x)$ for $2.5 < x < 3$.

Solution:

(a) For $f(x) = e^{-3x}$, we first compute the derivatives and find $f^{(n)}(x) = (-3)^n e^{-3x}$ and $f^{(n)}(2) = (-3)^n e^{-6}$. Then,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n e^{-6}}{n!}(x-2)^n$$

(b) Apply the ratio test to the series in part (a) to obtain

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} 3^{n+1} e^{-6} (x-2)^{n+1} n!}{(-1)^n 3^n e^{-6} (x-2)^n (n+1)!} \right| = 3 \cdot \lim_{n\to\infty} \left| \frac{x-2}{n+1} \right| = 0$$

Thus, the interval of convergence for the series is all real numbers and the radius of convergence is $\boxed{\infty}$.

(c) $T_2(x) = e^{-6} - 3e^{-6}(x-2) + \frac{9}{2}e^{-6}(x-2)^2$.

(d) The error term is $R_2(x) = \frac{f'''(z)}{3!}(x-2)^3$. Since $2.5 < x < 3$, the term $\left|(x-2)^3\right|$ has a maximum value when $x = 3$. Since the center $a = 2$ is outside of the range $2.5 < x < 3$, we need to consider z within the range $2 \le z \le 3$. With this in mind, the derivative $|f'''(z)| = 3^3 e^{-3z}$ has a maximum value when $z = 2$. It follows that

$$|R_2(x)| = \left| \frac{f'''(z)}{3!}(x-2)^3 \right| < \frac{3^3 e^{-3(2)}}{3!}(3-2)^3 = \frac{9}{2}e^{-6}.$$

5. (16 points) Consider the series

$$1 + x^2 + x^4 + x^6 + \cdots = \sum_{n=0}^{\infty} x^{2n},$$

for $|x| < 1$. In the following problems, your answer should not have sigma notation.

   (a) Find the sum of the series.

   (b) Find the sum of the series $\sum_{n=1}^{\infty} 2nx^{2n}$.

   (c) Find the sum of the series $\sum_{n=1}^{\infty} \frac{2n}{3^n}$.

Solution:

(a) This is a geometric series with first term $a = 1$ and ratio $r = x^2$. The sum of the series is $\frac{a}{1-r} = \frac{1}{1-x^2}$.

(b) From part (a) we have, for $-1 < x < 1$,

$$\sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2}$$

Differentiate both sides (term-by-term differentiation) to obtain

$$\sum_{n=1}^{\infty} 2nx^{2n-1} = \frac{2x}{(1-x^2)^2}$$

Then, multiply both sides by x to obtain

$$\sum_{n=1}^{\infty} 2nx^{2n} = \boxed{\frac{2x^2}{(1-x^2)^2}}$$

(c) Use the result from part (b) and let $x = \frac{1}{\sqrt{3}}$.

$$\sum_{n=1}^{\infty} \frac{2n}{3^n} = \sum_{n=1}^{\infty} 2n\left(\frac{1}{\sqrt{3}}\right)^{2n} = \sum_{n=1}^{\infty} 2nx^{2n}\Bigg|_{x=1/\sqrt{3}} = \frac{2x^2}{(1-x^2)^2}\Bigg|_{x=1/\sqrt{3}} = \frac{2/3}{(1-1/3)^2} = \boxed{\frac{3}{2}}.$$

6. (18 points) Consider the parametric equations for one arch of the cycloid given $x = a(t - \sin(t))$ and $y = a(1 - \cos(t))$, for $0 \leq t \leq 2\pi$ and a a fixed, positive constant.

(a) Find the area under one arch of the cycloid.

(b) Set up, but don't evaluate, the integral to find the length of one arch of the cycloid.

Solution:

(a) We have that $x'(t) = a(1 - \cos t)$. Then, in parametric form, the area between the curve and the x-axis is given by

$$\begin{aligned} A = \int_a^b yx'\,dt &= \int_0^{2\pi} a(1-\cos t)a(1-\cos t)\,dt \\ &= a^2\int_0^{2\pi} 1 - 2\cos t + \cos^2 t\,dt \\ &= a^2\int_0^{2\pi} 1 - 2\cos t + \frac{1}{2}(1+\cos 2t)\,dt \\ &= a^2\int_0^{2\pi} \frac{3}{2} - 2\cos t + \frac{1}{2}\cos 2t\,dt \\ &= a^2\left(\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t\right)\Bigg|_0^{2\pi} \\ &= \boxed{3a^2\pi.} \end{aligned}$$

(b) We have that

$$x'(t) = a(1-\cos t) \qquad \text{and} \qquad y'(t) = a\sin t.$$

Then, the arc length of an arch of the cycloid is given by

$$L = \int_a^b \sqrt{(x')^2 + (y')^2}\,dt = \int_0^{2\pi} \sqrt{(a(1-\cos t))^2 + (a\sin t)^2}\,dt$$

$$= \int_0^{2\pi} \sqrt{a^2(1 - 2\cos t + \cos^2 t + \sin^2 t)}\, dt$$

$$= \boxed{\int_0^{2\pi} a\sqrt{2 - 2\cos t}\, dt.}$$

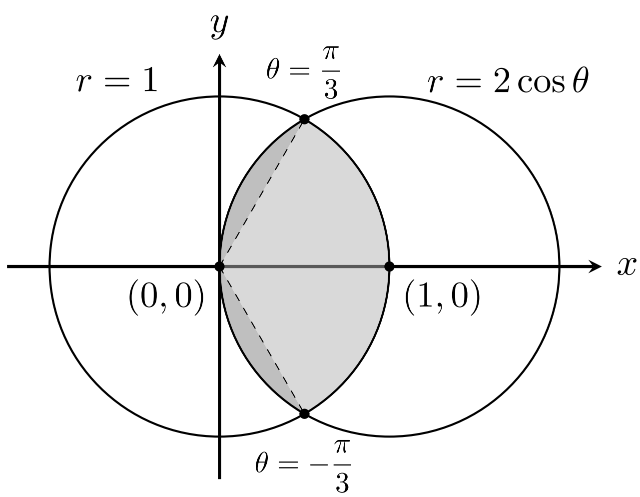
7. (16 points) Consider the two circles given by $r = 1$ and $r = 2\cos(\theta)$

   (a) Graph both circles in the xy-plane. Clearly label the center of each circle and all points of intersection.

   (b) Find the area of intersection of both circles, that is, find the area common to both circles.

Solution:

(a) The polar graph is given by



where the intersection points were found by setting the two equations equal to each other:

$$1 = 2\cos\theta \implies \cos\theta = \frac{1}{2} \implies \theta = -\frac{\pi}{3}, \frac{\pi}{3}.$$

(b) The shaded regions in part (a) show the area we are trying to compute. We can setup integrals to add the areas in the darker regions to the areas in the ligher region. Using symmetry, we find the area as

$$A = \int_{-\pi/2}^{-\pi/3} \frac{1}{2}(2\cos\theta)^2\, d\theta + \int_{-\pi/3}^{\pi/3} \frac{1}{2}(1)^2\, d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(2\cos\theta)^2\, d\theta$$

$$= 2\left(\int_0^{\pi/3} \frac{1}{2}(1)^2\, d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(2\cos\theta)^2\, d\theta\right) \qquad \text{by symmetry}$$

$$= \int_0^{\pi/3} d\theta + \int_{\pi/3}^{\pi/2} 4\cos^2\theta\, d\theta$$

$$= \int_0^{\pi/3} d\theta + \int_{\pi/3}^{\pi/2} 2(1 + \cos 2\theta)\theta\, d\theta$$

$$= \theta\Big|_0^{\pi/3} + 2\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{\pi/3}^{\pi/2}$$

$$= \left(\frac{\pi}{3} - 0\right) + 2\left(\frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) = \boxed{\frac{2\pi}{3} - \frac{\sqrt{3}}{2}.}$$