1. (54 pts) Consider the surface $x^2 - 10x + 4y^2 - 4z^2 + 25 = 0$.

   (a) Points $P(11, 0, 3)$ and $Q(5, -2, 2)$ lie on the surface.

   i. Find a vector of length 4 in the direction of $\mathbf{PQ}$.

   ii. Find symmetric equations for the line passing through P and Q.

   iii. Find an equation for the plane tangent to the surface at P.

   iv. Let $h(x, y, z) = y + xz$. Find the directional derivative of h at P in the direction toward Q.

   (b) Write the equation for the surface in standard form and identify the surface.

   (c) Classify the type of curves in the $x = 0$, $y = 0$, and $z = 0$ traces of the surface.

   (d) The path $\mathbf{r}(t) = \left\langle 4 + t^2,\ \ t,\ \ \frac{1}{2} + \frac{1}{2}t^2 \right\rangle$ lies on the surface. Let the temperature along the path be $T(x, y, z) = \dfrac{y}{x} - z$. Find dT/dt, the rate at which the temperature is changing, at $t = 2$.

Solution:

(a) i. $\mathbf{PQ} = \langle -6, -2, -1 \rangle$ and $|\mathbf{PQ}| = \sqrt{6^2 + 2^2 + 1^2} = \sqrt{41}$, so a unit vector in the same direction is $\frac{\mathbf{PQ}}{|\mathbf{PQ}|} = \frac{1}{\sqrt{41}} \langle -6, -2, -1 \rangle$. A vector in the direction of $\mathbf{PQ}$ of length 4 is $\boxed{\frac{4}{\sqrt{41}} \langle -6, -2, -1 \rangle}$.

ii. The line has direction $\mathbf{PQ} = \langle -6, -2, -1 \rangle$ and passes through $P(11, 0, 3)$, so a vector equation for the line is $\langle x, y, z \rangle = \langle 11, 0, 3 \rangle + t\langle -6, -2, -1 \rangle$. Parametric equations for the line are $x = 11 - 6t$, $y = -2t$, and $z = 3 - t$, and symmetric equations are

$$\boxed{\frac{x - 11}{-6} = \frac{y}{-2} = \frac{z - 3}{-1}}.$$

or

$$\boxed{\frac{x - 5}{-6} = \frac{y + 2}{-2} = \frac{z - 2}{-1}}$$

using $Q(5, -2, 2)$.

iii. Let $f(x, y, z) = x^2 - 10x + 4y^2 - 4z^2 + 25$. Then an equation for the tangent plane at P is

$$\nabla f(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\nabla f(11, 0, 3) \cdot \langle x - 11, y, z - 3 \rangle = 0.$$

Solve for ∇f at P:

$$\nabla f(x, y, z) = \langle 2x - 10, 8y, -8z \rangle$$

$$\nabla f(11, 0, 3) = \langle 12, 0, -24 \rangle,$$

so an equation for the tangent plane at P is

$$\boxed{12(x-11)-24(z-3)=0}$$

or

$$\boxed{x-2z=5}.$$

Alternate solution:

$$z=f(x,y)=\frac{1}{2}\sqrt{x^2-10x+4y^2+25}$$

for $z>0$. An equation for the tangent plane at P is

$$z=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$$
$$z=f(11,0)+f_x(11,0)(x-11)+f_y(11,0)y$$

Given

$$\nabla f=\frac{1}{2}\left\langle\frac{x-5}{\sqrt{x^2-10x+4y^2+25}},\frac{4y}{\sqrt{x^2-10x+4y^2+25}}\right\rangle$$
$$\nabla f(11,0)=\left\langle\frac{1}{2},0\right\rangle,$$

an equation for the tangent plane is

$$\boxed{z=3+\frac{1}{2}(x-11)}=\frac{1}{2}x-\frac{5}{2}.$$

iv.

$$h(x,y,z)=y+xz$$
$$\nabla h(x,y,z)=\langle z,1,x\rangle$$

The directional derivative of h at P in the direction of $\mathbf{PQ}$ equals

$$\nabla h(11,0,3)\cdot\frac{\mathbf{PQ}}{|\mathbf{PQ}|}=\langle 3,1,11\rangle\cdot\frac{\langle -6,-2,-1\rangle}{\sqrt{41}}$$
$$=\frac{-18-2-11}{\sqrt{41}}=\boxed{\frac{-31}{\sqrt{41}}}.$$

(b)

$$x^2-10x+4y^2-4z^2+25=0$$
$$\boxed{(x-5)^2+4y^2-4z^2=0}$$
$$\frac{(x-5)^2}{4}+y^2=z^2$$

The surface is a $\boxed{\text{cone}}$.

(c) The $x=0$ trace is $\frac{25}{4}+y^2=z^2$ which corresponds to a $\boxed{\text{hyperbola}}$.

The $y=0$ trace is $\frac{(x-5)^2}{4}=z^2$ which corresponds to the $\boxed{\text{two lines}}$ $2z=|x-5|$.

The $z=0$ trace is $\frac{(x-5)^2}{4}+y^2=0$ which corresponds to the $\boxed{\text{point}}$ $(5,0,0)$.

(d) Note that

$$\begin{aligned}\mathbf{r}(2) &= \left\langle 8, 2, \tfrac{5}{2} \right\rangle \\ \mathbf{r}'(t) &= \langle 2t, 1, t \rangle \\ \mathbf{r}'(2) &= \langle 4, 1, 2 \rangle\end{aligned}$$

and

$$\begin{aligned}\nabla T(x, y, z) &= \left\langle -\frac{y}{x^2}, \frac{1}{x}, -1 \right\rangle \\ \nabla T\left(8, 2, \tfrac{5}{2}\right) &= \left\langle -\frac{1}{32}, \frac{1}{8}, -1 \right\rangle.\end{aligned}$$

The rate at which the temperature is changing is

$$\begin{aligned}\frac{dT}{dt} &= \frac{dT}{dx}\frac{dx}{dt} + \frac{dT}{dy}\frac{dy}{dt} + \frac{dT}{dz}\frac{dz}{dt} \\ &= \nabla T \cdot \mathbf{r}'(t).\end{aligned}$$

At $t = 2$,

$$\begin{aligned}\left.\frac{dT}{dt}\right|_{t=2} &= \nabla T\left(8, 2, \tfrac{5}{2}\right) \cdot \mathbf{r}'(2) \\ &= \left\langle -\frac{1}{32}, \frac{1}{8}, -1 \right\rangle \cdot \langle 4, 1, 2 \rangle \\ &= -\frac{1}{8} + \frac{1}{8} - 2 \\ &= \boxed{-2}.\end{aligned}$$

2. (18 pts) A joint probability density function for random variables X and Y is

$$f(x, y) = \frac{1}{2\pi} e^{\left(-x^2 - y^2\right)/2}$$

for all real x, y. Consider the probability $P\left(Y \geq 0 \text{ and } X^2 + Y^2 \geq 16\right)$.

(a) Set up a double integral in polar coordinates to compute the probability.

(b) Evaluate the integral.

Solution:

(a) Let R be the region with $y \geq 0$ and $x^2 + y^2 \geq 16$, which consists of all points in the upper half of the xy-plane exterior to a circle of radius 4 centered at the origin. In polar coordinates, the region corresponds to $0 \leq \theta \leq \pi$ and $r \geq 4$. Substituting $r^2 = x^2 + y^2$ and the Jacobian r gives

$$P\left(Y \geq 0 \text{ and } X^2 + Y^2 \geq 16\right) = \iint_R \frac{1}{2\pi} e^{\left(-x^2 - y^2\right)/2}\, dA = \int_0^{\pi} \int_4^{\infty} \frac{1}{2\pi} r e^{-r^2/2}\, dr\, d\theta.$$

(b)

$$\int_0^\pi \int_4^\infty \frac{1}{2\pi} r e^{-r^2/2}\, dr\, d\theta = \left(\int_0^\pi d\theta\right) \int_4^\infty \underbrace{\frac{1}{2\pi} r e^{-r^2/2}\, dr}_{u=r^2/2,\, du=r\,dr}$$
$$= \left(\int_0^\pi d\theta\right) \int_8^\infty \frac{1}{2\pi} e^{-u}\, du$$
$$= \left[\theta\right]_0^\pi \left[-\frac{1}{2\pi} e^{-u}\right]_8^\infty$$
$$= \pi \left(-\frac{1}{2\pi}\right) \lim_{t\to\infty} \left(e^{-t} - e^{-8}\right)$$
$$= \boxed{\frac{1}{2} e^{-8}}$$

3. (28 pts) Region E lies in the **first octant**, below a sphere of radius 3 centered at the origin, and above the surface $z = 3 - \sqrt{x^2+y^2}$.

   (a) Sketch and shade a cross-section of the region in the rz-plane (i.e., a half-plane of constant θ).

   (b) Set up (but do not evaluate) an integral to find the volume of the region using

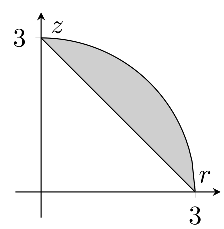
      i. rectangular coordinates in the order $dz\,dy\,dx$

      ii. cylindrical coordinates in the order $dz\,dr\,d\theta$

      iii. spherical coordinates in the order $d\rho\,d\phi\,d\theta$.

Solution:

(a) Region E lies between a sphere of radius 3 and an inverted cone. A cross-section of the region in the rz-plane is shown below.



(b) An equation for the sphere is

$$x^2 + y^2 + z^2 = 9 \implies z = \sqrt{9 - x^2 - y^2}.$$

The projection of E onto the xy-plane is a quarter-circular region of radius 3 in the first quadrant. The circle has equation $x^2 + y^2 = 9$, so the quarter-circle is defined by $y = \sqrt{9 - x^2}$.

   i. In rectangular coordinates, the projected region in the xy-plane corresponds to $0 \le x \le 3$ and $0 \le y \le \sqrt{9 - x^2}$, so the volume integral is

$$\int_{x=0}^{3} \int_{y=0}^{\sqrt{9-x^2}} \int_{z=3-\sqrt{x^2+y^2}}^{\sqrt{9-x^2-y^2}} dz\, dy\, dx.$$

ii. In polar coordinates, the projected region corresponds to θ in $[0, \frac{\pi}{2}]$ and r in $[0, 3]$. Substituting $r^2 = x^2 + y^2$ and the Jacobian r, the volume integral is

$$\int_{\theta=0}^{\pi/2} \int_{r=0}^{3} \int_{z=3-r}^{\sqrt{9-r^2}} r \, dz \, dr \, d\theta.$$

iii. In spherical coordinates, region E in the first octant corresponds to θ in $[0, \frac{\pi}{2}]$ and ϕ in $[0, \frac{\pi}{2}]$. The variable ρ extends from the cone $z = 3 - r$ to the sphere $\rho = 3$. Substituting the identities $z = \rho \cos \phi$ and $r = \rho \sin \phi$, then solving for ρ, gives a lower bound of

$$z = 3 - r \implies \rho \cos \phi = 3 - \rho \sin \phi \implies \rho = \frac{3}{\cos \phi + \sin \phi}.$$

Inserting the Jacobian $\rho^2 \sin \phi$, the volume integral is

$$\int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \int_{\rho=3/(\cos \phi + \sin \phi)}^{3} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

4. (14 pts) Let A and B be symmetric $n \times n$ matrices. Prove that $(AB)^T = BA$ in two ways:

(a) Using properties of matrix transpose and multiplication.

(b) Examining the (i, j) entries of $(AB)^T$ and BA.

Solution:

(a) Because A and B are symmetric matrices, $A = A^T$ and $B = B^T$. Then

$$(AB)^T = B^T A^T = BA$$

using the transpose property of matrix product.

(b)

$$\begin{aligned}
(i, j) \text{ entry of } (AB)^T &= (j, i) \text{ entry of } AB && \text{(definition of matrix transpose)} \\
&= (\text{row } j \text{ of } A) \cdot (\text{col } i \text{ of } B) && \text{(definition of matrix mult)} \\
&= (\text{col } j \text{ of } A) \cdot (\text{row } i \text{ of } B) && \text{(symmetric matrix property)} \\
&= (\text{row } i \text{ of } B) \cdot (\text{col } j \text{ of } A) && \text{(commutative property of dot product)} \\
&= (i, j) \text{ entry of } BA && \text{(definition of matrix mult)}
\end{aligned}$$

5. (14 pts) Use row reduction to find the inverse of the matrix.

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 1 & -2 & 0 \end{bmatrix}$$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & -2 & -4 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & -8 & -1 & 2 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{8} & -\frac{1}{4} & -\frac{1}{8} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{8} & -\frac{1}{4} & -\frac{1}{8} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{8} & -\frac{1}{4} & -\frac{1}{8} \end{array}\right]$$

The inverse is

$$\begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{8} & -\frac{1}{4} & -\frac{1}{8} \end{bmatrix}.$$

6. (10 pts) A homogeneous system has 2 equations and 5 variables x_1, x_2, x_3, x_4, x_5. The fundamental solutions for the system are

$$b(-5,1,0,0,0) + d(2,0,0,1,0) + e(0,0,3,0,1)$$

where $x_2 = b$, $x_4 = d$, and $x_5 = e$, for real numbers b, d, and e. Find the reduced row echelon form (RREF) augmented matrix that represents the system.

Solution:

The homogeneous system has independent variables $x_2 = b$, $x_4 = d$, and $x_5 = e$. Given the fundamental solutions $b(-5,1,0,0,0) + d(2,0,0,1,0) + e(0,0,3,0,1)$, the complete solution set is

$$\{(-5b + 2d, b, 3e, d, e) \mid b, d, e \text{ in } \mathbb{R}\}.$$

The solution set corresponds to the equations

$$\begin{aligned} x_1 = -5b + 2d &\implies x_1 + 5b - 2d = 0 \implies x_1 + 5x_2 - 2x_4 = 0 \\ x_3 = 3e &\implies x_3 - 3e = 0 \implies x_3 - 3x_5 = 0. \end{aligned}$$

An augmented matrix that representing this homogeneous system is

$$\left[\begin{array}{ccccc|c} 1 & 5 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 \end{array}\right]$$

which is in reduced row echelon form.

7. (12 pts) Suppose we wish to find a quadratic least-squares polynomial for the points

$$(-1,-2),\quad (0,0),\quad (1,0),\quad (2,-1).$$

Set up an augmented matrix representing the linear system that can be solved to determine the quadratic polynomial. (Do not solve the system.)

Solution: Let the quadratic function be $y = c_0 + c_1 x + c_2 x^2$. We wish to find constants c_0, c_1, and c_2 that will minimize the least-squares error. Let

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

We wish to solve $\mathbf{A}^T\mathbf{A}\mathbf{v} = \mathbf{A}^T\mathbf{b}$. Given $\mathbf{A}^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix}$, the augmented matrix that will lead to the solution is

$$\left[\mathbf{A}^T\mathbf{A} \mid \mathbf{A}^T\mathbf{b}\right] = \left[\begin{array}{ccc|c} 4 & 2 & 6 & -3 \\ 2 & 6 & 8 & 0 \\ 6 & 8 & 18 & -6 \end{array}\right].$$