1. [2360/121125 (24 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a two-column table (letter - answer) completely separate from any work you do to arrive at the answer.

   (a) The differential equation $y' = t(y-2)^2$ has two equilibrium solutions.

   (b) If $|\mathbf{A}| = 0$, where $\mathbf{A}$ is an $n \times n$ matrix, the column space of $\mathbf{A}$ is a subspace of $\mathbb{R}^n$.

   (c) The system $\begin{array}{l} x' = x^2 + y^2 - 8 \\ y' = x - y \end{array}$ has no equilibrium solutions.

   (d) The set of all $n \times n$ diagonal matrices is not a subspace of $\mathbb{M}_{n\times n}$.

   (e) The harmonic oscillator governed by $5\ddot{x} + 45x = \sin 3t$ is in resonance.

   (f) Picard's Theorem guarantees that a unique solution to the initial value problem $y' = y + \text{step}(t-1),\ y(1) = 2$ does not exist.

   (g) If $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4\}$ is a set of vectors in $\mathbb{P}_3$ such that $\vec{\mathbf{v}}_2 = 3\vec{\mathbf{v}}_3 - \vec{\mathbf{v}}_4 + 7\vec{\mathbf{v}}_1$, then $\mathbb{P}_3 \neq \text{span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4\}$.

   (h) $y(t) = c_1 + c_2 \ln t$ where $c_1, c_2 \in \mathbb{R}$ is the general solution of $ty'' + y' = 0$ on the interval $[1, \infty)$.

   SOLUTION:

   (a) **FALSE** $y = 2$ is the only equilibrium solution.

   (b) **TRUE** The columns of $\mathbf{A}$ are linearly dependent and thus cannot span $\mathbb{R}^n$. However, their span forms a subspace of $\mathbb{R}^n$.

   (c) **FALSE** $(2, 2)$ and $(-2, -2)$ are equilibrium solutions.

   (d) **FALSE** The set is a subspace since it is closed under vector addition and scalar multiplication.

   (e) **TRUE** $\sqrt{\frac{45}{5}} = 3$ which equals the frequency of the forcing function. Note that the guess for the particular solution when using the method of undetermined coefficients is $x_p = At\sin 3t + Bt\cos t$.

   (f) **FALSE** $y + \text{step}(t-1)$ is not continuous at $(1, 2)$ so Picard's Theorem guarantees nothing.

   (g) **TRUE** The 4 vectors are linearly dependent and thus cannot be a basis.

   (h) **TRUE** $t(c_1 + c_2\ln t)'' + (c_1 + c_2 \ln t)' = t\left(0 - \frac{1}{t^2}\right) + \frac{1}{t} = -\frac{1}{t} + \frac{1}{t} = 0$

■

2. [2360/121125 (35 pts)] Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$. Hint: This problem involves no fractions.

   (a) (6 pts) Let $\vec{\mathbf{b}} = \begin{bmatrix} 2 & k & 3 \end{bmatrix}^{\mathrm{T}}$. Are there any real values of k for which $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent? If so, find them. If not, explain why not.

   (b) (4 pts) Find a basis for the solution space of $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$. Feel free to use your work from part (a), making sure it is correct before doing so.

   (c) (3 pts) Is $\mathbf{A}$ singular? Justify your answer.

   (d) (4 pts) Verify that $\mathbf{A}$ has eigenvalues $\lambda = 0, 2$ and state the algebraic multiplicity of each.

   (e) (8 pts) Find the eigenvectors for each eigenvalue in part (d) and state the geometric multiplicity of each one. Also, give the dimension of the eigenspace of each eigenvalue.

   (f) (10 pts) Solve the IVP $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} 0 & 4 & 2 \end{bmatrix}^{\mathrm{T}}$. Write your final answer as a single vector. Hint: Most of the work has already been completed in the previous parts. Make sure it is correct prior to using it!

   SOLUTION:

(a) Yes. $k = 2$. Justification:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & k \\ 0 & 1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & -2+k \\ 0 & 1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -2+k \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -2+k \end{array}\right]$$

and with $k = 2$ we have the RREF $\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(b) For the homogeneous system we have, using the work from part (a),

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{x}} = \begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix},\ t \in \mathbb{R} \implies \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\} \text{ is a basis for the solution space}$$

(c) Yes. The RREF contains a row of zeros. Another justification is that $|\mathbf{A}| = 0$. There are other justifications.

(d)

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)(-1)^{3+3} \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)[(1-\lambda)(1-\lambda) - 1] = (2-\lambda)(-2\lambda + \lambda^2) = -\lambda(\lambda-2)^2 = 0$$

Thus, $\lambda = 0$ with algebraic multiplicity of 1 and $\lambda = 2$ with algebraic multiplicity of 2.

(e) For $\lambda = 0$, solve $(\mathbf{A} - 0\mathbf{I})\,\vec{\mathbf{v}} = \vec{\mathbf{0}}$. This is the same as part (b). An eigenvector is $\vec{\mathbf{v}} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ so that the geometric multiplicity of $\lambda = 0$ is 1 as is the dimension of the eigenspace $\mathbb{E}_{\lambda=0}$.

For $\lambda = 2$, solve $(\mathbf{A} - 2\mathbf{I})\,\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The eigenvalue $\lambda = 2$ has geometric multiplicity of 1 which is the dimension of the eigenspace $\mathbb{E}_{\lambda=1}$.

(f) Find a generalized eigenvector for the defective eigenvalue $\lambda = 2$. Solve $(\mathbf{A} - 2\mathbf{I})\,\vec{\mathbf{u}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The general solution is

$$\vec{\mathbf{x}}(t) = c_1 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{2t} \left(t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right)$$

Applying the initial condition yields

$$\left.\begin{aligned} 2c_1 + c_3 &= 0 \\ -2c_1 + c_3 &= 4 \\ c_1 + c_2 &= 2 \end{aligned}\right\} \implies \begin{bmatrix} 2 & 0 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

There are several ways to solve this system: RREF, matrix inverse, Cramer's Rule. Using the latter:

$$c_1 = \frac{\begin{bmatrix} 0 & 0 & 1 \\ 4 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}}{\begin{bmatrix} 2 & 0 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}} = \frac{4}{-4} = -1 \qquad c_2 = \frac{\begin{bmatrix} 2 & 0 & 1 \\ -2 & 4 & 1 \\ 1 & 2 & 0 \end{bmatrix}}{\begin{bmatrix} 2 & 0 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}} = \frac{-12}{-4} = 3 \qquad c_3 = \frac{\begin{bmatrix} 2 & 0 & 0 \\ -2 & 0 & 4 \\ 1 & 1 & 0 \end{bmatrix}}{\begin{bmatrix} 2 & 0 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}} = \frac{-8}{-4} = 2$$

Thus,

$$\vec{\mathbf{x}}(t) = -1\begin{bmatrix}2\\-2\\1\end{bmatrix} + 3e^{2t}\begin{bmatrix}0\\0\\1\end{bmatrix} + 2e^{2t}\left(t\begin{bmatrix}0\\0\\1\end{bmatrix} + \begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2e^{2t}-2\\2e^{2t}+2\\(2t+3)e^{2t}-1\end{bmatrix}$$

Remark concerning the linear system for the coefficients: the RREF is $\left[\begin{array}{ccc|c}1&0&0&-1\\0&1&0&3\\0&0&1&2\end{array}\right]$ and the inverse is $\begin{bmatrix}\frac{1}{4}&-\frac{1}{4}&0\\-\frac{1}{4}&\frac{1}{4}&1\\\frac{1}{2}&\frac{1}{2}&0\end{bmatrix}$

■

3. [2360/121125 (10 pts)] If water is drained from a vertical cylindrical tank by opening a valve at the base of the tank, the rate of change at which the water level $[y(t),\text{ in feet}]$ drops is proportional to the square root of the water's depth, that is $y' = -k\sqrt{y}$. If t is measured in minutes and $k = 0.1$, how long will it take the tank to drain assuming the water is 9 feet deep to start with?

SOLUTION:

$$\int y^{-1/2}\,\mathrm{d}y = \int -\frac{1}{10}\,\mathrm{d}t$$

$$2y^{1/2} = -\frac{t}{10} + C \qquad \text{apply the intial condition}$$

$$2\sqrt{9} = 0 + C \implies C = 6 \implies 2\sqrt{y} = 6 - \frac{t}{10}$$

The tank is considered drained when the depth, y, is 0. Hence,

$$0 = 6 - \frac{t}{10} \implies t = 60$$

The tank is drained in 60 minutes or 1 hour. ■

4. [2360/121125 (25 pts)] Consider the initial value problem $y'' + 4y' + 20y = 40 + \delta(t-2),\ y(0) = 1,\ y'(0) = 0$.

   (a) (5 pts) Could this initial value problem govern the motion of an harmonic oscillator? If so, (i) what type of damping is present; (ii) describe the initial state of the oscillator, assuming it is oriented horizontally. If not, explain why not.

   (b) (20 pts) Use Laplace transforms to solve the IVP.

SOLUTION:

(a) Yes, it can describe an harmonic oscillator since all of the coefficients on the left are positive. Since $4^2 - 4(1)(20) = -64 < 0$ the oscillator is underdamped. Initially, the mass is at rest one unit to the right of equilibrium position.

(b) Taking the Laplace Transform of both sides yields

$$s^2Y(s) - sy(0) - y'(0) + 4[sY(s) - y(0)] + 20Y(s) = \frac{40}{s} + e^{-2s}$$

$$\left(s^2 + 4s + 20\right)Y(s) = \frac{40}{s} + e^{-2s} + (s+4)y(0) + y'(0)$$

$$Y(s) = \frac{40}{s(s^2+4s+20)} + \frac{e^{-2s}}{s^2+4s+20} + \frac{s+4}{s^2+4s+20}$$

Partial fraction decomposition of the first term:

$$\frac{40}{s(s^2+4s+20)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+20}$$

$$40 = A(s^2+4s+20) + (Bs+C)s$$

$$s = 0 : 40 = A(20) \implies A = 2$$

$$s = 1 : 40 = 2(25) + B + C \implies B + C = -10 \qquad (1)$$

$$s = -1 : 40 = 2(17) + B - C \implies B - C = 6 \qquad (2)$$

$$\text{adding (1) and (2) yields} \quad 2B = -4 \implies B = -2 \implies C = -8$$

Using this decomposition we have

$$Y(s) = \frac{2}{s} - \frac{2(s+4)}{s^2+4s+20} + \frac{e^{-2s}}{s^2+4s+20} + \frac{s+4}{s^2+4s+20} = \frac{2}{s} - \frac{s+4}{s^2+4s+20} + \frac{e^{-2s}}{s^2+4s+20}$$

$$= \frac{2}{s} - \frac{s+2}{(s+2)^2+16} - \frac{1}{2}\frac{4}{(s+2)^2+16} + \frac{1}{4}\frac{4e^{-2s}}{(s+2)^2+16}$$

$$y(t) = 2 - e^{-2t}\cos 4t - \frac{1}{2}e^{-2t}\sin 4t + \frac{1}{4}e^{-2(t-2)}\sin 4(t-2)\,\text{step}(t-2)$$

■

5. [2360/121125 (18 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ -1 & k \end{bmatrix}$ where $k \in \mathbb{R}$. In your bluebook, write a column with the letters a-f separate from your work. Next to each letter, give the value(s) of k, if any, that will make the linear system $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ possess the stated behavior. Write NONE as appropriate. No work need be shown and no partial credit is available.

(a) nonisolated fixed points (b) repeated positive eigenvalues (c) attracting spiral (d) center (e) repelling node (f) saddle

SOLUTION:

$$\text{Tr}\,\mathbf{A} = k+3; \quad |\mathbf{A}| = 3k+4; \quad (\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| = (k+3)^2 - 4(3k+4) = k^2-6k-7 = (k-7)(k+1)$$

(a) $|\mathbf{A}| = 3k+4 = 0 \implies \boxed{k = -\frac{4}{3}}$

(b) For repeated eigenvalues: $(\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| = (k-7)(k+1) = 0 \implies k = -1, 7$; For positive eigenvalues: $\text{Tr}\,\mathbf{A} = k+3 > 0 \implies k > -3 \implies \boxed{k = -1, 7}$

(c) $(\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| = (k-7)(k+1) < 0 \implies -1 < k < 7$ and $\text{Tr}\,\mathbf{A} = k+3 < 0 \implies k < -3 \implies \boxed{\text{NONE}}$

(d) $\text{Tr}\,\mathbf{A} = k+3 = 0 \implies k = -3$ and $|\mathbf{A}| = 3k+4 > 0 \implies k > -\frac{4}{3} \implies \boxed{\text{NONE}}$

(e) $(\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| = (k-7)(k+1) > 0 \implies k < -1$ or $k > 7$ and $\text{Tr}\,\mathbf{A} = k+3 > 0 \implies k > -3$ and $|\mathbf{A}| = 3k+4 > 0 \implies k > -\frac{4}{3} \implies \boxed{-\frac{4}{3} < k < -1 \text{ or } k > 7}$

(f) $|\mathbf{A}| = 3k+4 < 0 \implies \boxed{k < -\frac{4}{3}}$

■

6. [2360/121125 (20 pts)] The charge, $q(t)$, in a certain RLC-circuit is governed by the initial value problem

$$2\ddot{q} + 12\dot{q} + 10q = 16e^{-t} + 100, \; q(0) = \dot{q}(0) = 14$$

(a) (17 pts) Use the method of undetermined coefficients to find the charge in the circuit. No points awarded for using another method.

(b) (3 pts) Does the charge approach a steady state? If so, find it. If not, explain why not.

SOLUTION:

(a)

$$2r^2 + 12r + 10 = (2r+10)(r+1) = 0 \implies r = -5, -1 \implies q_h(t) = c_1e^{-5t} + c_2e^{-t}$$

For the particular solution the guess is $q_p(t) = A + Bte^{-t}$. Then

$$2\ddot{q}_p + 12\dot{q}_p + 10q_p = 2\left(Bte^{-t} - 2Be^{-t}\right) + 12\left(-Bte^{-t} + Be^{-t}\right) + 10\left(A + Bte^{-t}\right) = 8Be^{-t} + 10A = 16e^{-t} + 100$$

From this $A = 10$ and $B = 2$ and $q_p = 2te^{-t} + 10$. Applying the initial conditions to $q = q_h + q_p = c_1e^{-5t} + c_2e^{-t} + 2te^{-t} + 10$ gives

$$q(0) = c_1 + c_2 + 10 = 14$$

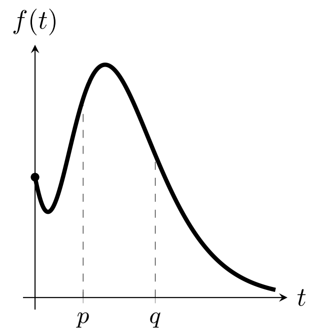
$$\dot{q}(t) = -5c_1e^{-5t} - c_2e^{-t} - 2te^{-t} + 2e^{-t} \implies \dot{q}(0) = -5c_1 - c_2 + 2 = 14$$

$$\left.\begin{aligned} c_1 + c_2 &= 4 \\ -5c_1 - c_2 &= 12 \end{aligned}\right\} \implies -4c_1 = 16 \implies c_1 = -4 \implies c_2 = 8$$

The charge in the circuit is thus $q(t) = 8e^{-t} - 4e^{-5t} + 2te^{-t} + 10$.

(b) Yes, the steady state charge is 10, obtained from evaluating $\lim_{t\to\infty} q(t) = 10$ (note that l'Hôpital's Rule was implicitly used).

7. [2360/121125 (18 pts)] Consider the graph of the function $f(t)$ shown here.



Match the graphs to the functions below. No work need be shown and no partial credit will be awarded. Your final answer should appear as a 6×2 matrix. The first column will be the letters a-f and the second column will contain the Roman numeral of the graph matching the correct function given in (a)-(f). Write NONE if none of the given graphs match the given functions. Note that $F(s) = \mathscr{L}\{f(t)\}$.

(a) $f(t-p)[\text{step}(t-p) - \text{step}(t-q)]$

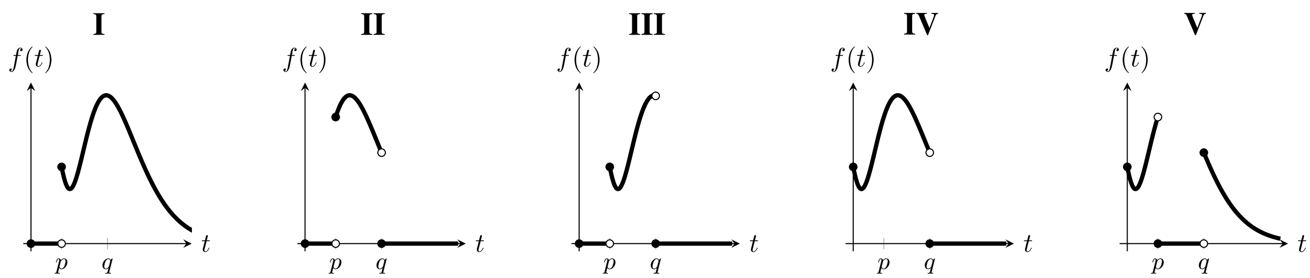
(b) $\mathscr{L}^{-1}\left\{e^{-qs}\mathscr{L}\left\{f(t+q)\right\}\right\}$

(c) $\mathscr{L}^{-1}\left\{e^{-ps}F(s)\right\}$

(d) $f(t) - f(t)\,\text{step}(t-q)$

(e) $f(t)\,[\text{step}(t-p) - \text{step}(t-q)]$

(f) $f(t)[1 - \text{step}(t-p) + \text{step}(t-q)]$



SOLUTION:

(a) **III**

(b) **NONE** $f(t)\text{step}(t-q)$

(c) **I** $f(t-p)\text{step}(t-p)$

(d) **IV**

(e) **II**

(f) **V**