- This exam is worth 150 points and has 7 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"× 11" crib sheet with writing on two sides.

0. At the top of the page containing your solution to problem 1, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/121125 (24 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a two-column table (letter - answer) completely separate from any work you do to arrive at the answer.

   (a) The differential equation $y' = t(y-2)^2$ has two equilibrium solutions.

   (b) If $|\mathbf{A}| = 0$, where $\mathbf{A}$ is an $n \times n$ matrix, the column space of $\mathbf{A}$ is a subspace of $\mathbb{R}^n$.

   (c) The system $\begin{aligned} x' &= x^2 + y^2 - 8 \\ y' &= x - y \end{aligned}$ has no equilibrium solutions.

   (d) The set of all $n \times n$ diagonal matrices is not a subspace of $\mathbb{M}_{n\times n}$.

   (e) The harmonic oscillator governed by $5\ddot{x} + 45x = \sin 3t$ is in resonance.

   (f) Picard's Theorem guarantees that a unique solution to the initial value problem $y' = y + \text{step}(t-1),\ y(1) = 2$ does not exist.

   (g) If $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4\}$ is a set of vectors in $\mathbb{P}_3$ such that $\vec{\mathbf{v}}_2 = 3\vec{\mathbf{v}}_3 - \vec{\mathbf{v}}_4 + 7\vec{\mathbf{v}}_1$, then $\mathbb{P}_3 \neq \text{span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4\}$.

   (h) $y(t) = c_1 + c_2 \ln t$ where $c_1, c_2 \in \mathbb{R}$ is the general solution of $ty'' + y' = 0$ on the interval $[1, \infty)$.

2. [2360/121125 (35 pts)] Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$. Hint: This problem involves no fractions.

   (a) (6 pts) Let $\vec{\mathbf{b}} = \begin{bmatrix} 2 & k & 3 \end{bmatrix}^{\mathrm{T}}$. Are there any real values of k for which $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent? If so, find them. If not, explain why not.

   (b) (4 pts) Find a basis for the solution space of $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$. Feel free to use your work from part (a), making sure it is correct before doing so.

   (c) (3 pts) Is $\mathbf{A}$ singular? Justify your answer.

   (d) (4 pts) Verify that $\mathbf{A}$ has eigenvalues $\lambda = 0, 2$ and state the algebraic multiplicity of each.

   (e) (8 pts) Find the eigenvectors for each eigenvalue in part (d) and state the geometric multiplicity of each one. Also, give the dimension of the eigenspace of each eigenvalue.

   (f) (10 pts) Solve the IVP $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} 0 & 4 & 2 \end{bmatrix}^{\mathrm{T}}$. Write your final answer as a single vector. Hint: Most of the work has already been completed in the previous parts. Make sure it is correct prior to using it!

3. [2360/121125 (10 pts)] If water is drained from a vertical cylindrical tank by opening a valve at the base of the tank, the rate of change at which the water level $[y(t),$ in feet$]$ drops is proportional to the square root of the water's depth, that is $y' = -k\sqrt{y}$. If t is measured in minutes and $k = 0.1$, how long will it take the tank to drain assuming the water is 9 feet deep to start with?

4. [2360/121125 (25 pts)] Consider the initial value problem $y'' + 4y' + 20y = 40 + \delta(t-2),\ y(0) = 1,\ y'(0) = 0$.

   (a) (5 pts) Could this initial value problem govern the motion of an harmonic oscillator? If so, (i) what type of damping is present; (ii) describe the initial state of the oscillator, assuming it is oriented horizontally. If not, explain why not.

   (b) (20 pts) Use Laplace transforms to solve the IVP.

MORE PROBLEMS AND LAPLACE TRANSFORM TABLE BELOW/ON REVERSE

5. [2360/121125 (18 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ -1 & k \end{bmatrix}$ where $k \in \mathbb{R}$. In your bluebook, write a column with the letters a-f separate from your work. Next to each letter, give the value(s) of k, if any, that will make the linear system $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ possess the stated behavior. Write NONE as appropriate. No work need be shown and no partial credit is available.

(a) nonisolated fixed points (b) repeated positive eigenvalues (c) attracting spiral (d) center (e) repelling node (f) saddle

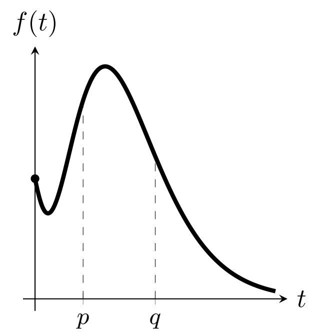
6. [2360/121125 (20 pts)] The charge, $q(t)$, in a certain RLC-circuit is governed by the initial value problem

$$2\ddot{q} + 12\dot{q} + 10q = 16e^{-t} + 100, \; q(0) = \dot{q}(0) = 14$$

(a) (17 pts) Use the method of undetermined coefficients to find the charge in the circuit. No points awarded for using another method.

(b) (3 pts) Does the charge approach a steady state? If so, find it. If not, explain why not.

7. [2360/121125 (18 pts)] Consider the graph of the function $f(t)$ shown here.

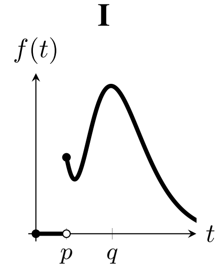


Match the graphs to the functions below. No work need be shown and no partial credit will be awarded. Your final answer should appear as a 6×2 matrix. The first column will be the letters a-f and the second column will contain the Roman numeral of the graph matching the correct function given in (a)-(f). Write NONE if none of the given graphs match the given functions. Note that $F(s) = \mathscr{L}\{f(t)\}$.

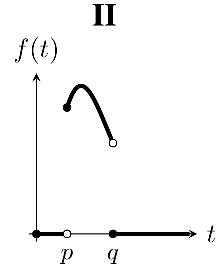
(a) $f(t-p)[\text{step}(t-p) - \text{step}(t-q)]$ (b) $\mathscr{L}^{-1}\left\{e^{-qs}\mathscr{L}\left\{f(t+q)\right\}\right\}$ (c) $\mathscr{L}^{-1}\left\{e^{-ps}F(s)\right\}$

(d) $f(t) - f(t)\,\text{step}(t-q)$ (e) $f(t)\,[\text{step}(t-p) - \text{step}(t-q)]$ (f) $f(t)[1 - \text{step}(t-p) + \text{step}(t-q)]$

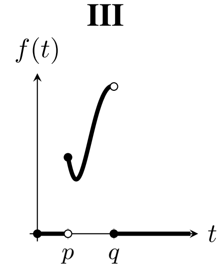
I



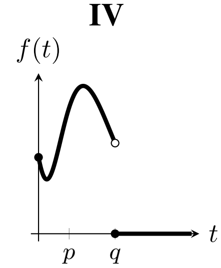
II



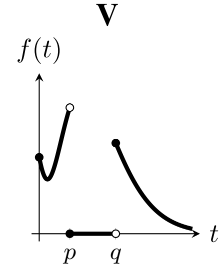
III



IV



V



Short table of Laplace Transforms: $\mathscr{L}\left\{f(t)\right\} = F(s) \equiv \displaystyle\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\left\{\cosh bt\right\} = \frac{s}{s^2-b^2} \qquad \mathscr{L}\left\{\sinh bt\right\} = \frac{b}{s^2-b^2}$$

$$\mathscr{L}\left\{t^n f(t)\right\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\left\{\delta(t-c)\right\} = e^{-cs}$$

$$\mathscr{L}\left\{tf'(t)\right\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\left\{f(t-c)\,\text{step}(t-c)\right\} = e^{-cs}F(s) \qquad \mathscr{L}\left\{f(t)\,\text{step}(t-c)\right\} = e^{-cs}\mathscr{L}\left\{f(t+c)\right\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$