1. [2350/121125 (24 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a two-column table (letter - answer) completely separate from any work you do to arrive at the answer.

   (a) $\mathbf{r}(u,v) = \langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle, 0 \le u \le \pi, 0 \le v \le \pi/2$ parameterizes the first octant portion of the sphere of radius $a > 0$ with center at $(0,0,0)$.

   (b) The first order Taylor polynomial of $f(x,y) = e^{-x^2-y^2}$ centered at the origin is the plane $T_1(x,y) = 1$.

   (c) If $u = xy$ and $v = xy^2$, the Jacobian used in the change of variables theorem for double integrals is v.

   (d) There exists a point in $\mathbb{R}^3$ where the plane $x+y+z=1$ and the line with symmetric equations $\frac{x}{2} = -y = -z$ intersect.

   (e) $f(x,y) = \dfrac{xy}{x^2+y^2}$ is continuous on its domain.

   (f) Suppose you are traveling along the differentiable path $\mathbf{r}(s)$ in $\mathbb{R}^2$, where s is the arc length parameter. At the point (a,b), which is not a critical point of the differentiable function $f(x,y)$, the directional derivative equals 0. Then you are instantaneously traveling in the direction of the gradient vector.

   (g) For c a real number, $cx^2 + y^2 + z^2 = 1$ describes either an ellipsoid, a sphere or hyperboloid of one sheet.

   (h) Let $f(x,y) = x^2 + xy + y^2$. For all (x,y) in $\mathbb{R}^2$, the rate of change in the y-direction of the slope of the tangent line in the x-direction is 1.

   SOLUTION:

   (a) **FALSE** The first octant portion requires $0 \le u \le \pi/2$.

   (b) **TRUE** $f_x = -2xe^{-x^2-y^2} \implies f_x(0,0) = 0$; $f_y = -2ye^{-x^2-y^2} \implies f_y(0,0) = 0$; $f(0,0) = 1$. Therefore, $T_1(x,y) = 1$.

   (c) **FALSE** $x = u^2/v, y = v/u$ and

   $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2u}{v} & -\frac{u^2}{v^2} \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{1}{v}$$

   (d) **FALSE** The normal vector to the plane is $\mathbf{n} = \langle 1,1,1\rangle$ and the direction vector of the line is $\mathbf{v} = \langle 2,-1,-1\rangle$ with $\mathbf{n}\cdot\mathbf{v} = 0$, implying that the line and plane are parallel and thus never intersect.

   (e) **TRUE** The function is a rational function, all of which are continuous on their domains. The domain of $f(x,y)$ here is $\mathbb{R}^2 - \{(0,0)\}$, that is, all of $\mathbb{R}^2$ except the origin.

   (f) **FALSE** You are instantaneously traveling in a direction orthogonal to the gradient vector, that is, along a level curve. This follows from the fact that neither $\mathbf{r}'(s)$ nor ∇f are the zero vector at the point so their dot product, the directional derivative, can be zero only if the vectors are orthogonal.

   (g) **FALSE** If $c = 0$, the equation describes a cylinder.

   (h) **TRUE** $f_{xy} = f_{yx} = 1$.

   ■

2. [2350/121125 (8 pts)] The cost, C, of a trip that is L miles long, driving a car that gets m miles per gallon, with gas costs of p dollars per gallon is $C = Lp/m$ dollars. You are planning a trip of $L = 100$ miles in a car whose gas mileage is $m = 40$ miles per gallon with the price of gas at $p = \$2.00$ per gallon. Use differentials to estimate the change in the trip's cost if you get detoured an extra 20 miles which lowers your gas mileage by 5 miles per gallon and requires you to pay $\$0.10$ more per gallon of gas.

SOLUTION:

$$\mathrm{d}C = \frac{\partial C}{\partial L}\,\mathrm{d}L + \frac{\partial C}{\partial m}\,\mathrm{d}m + \frac{\partial C}{\partial p}\,\mathrm{d}p$$

$$\begin{aligned}\mathrm{d}C &= \frac{p}{m}\,\mathrm{d}L - \frac{Lp}{m^2}\,\mathrm{d}m + \frac{L}{m}\,\mathrm{d}p \\ &= \frac{2}{40}(20) - \frac{(100)(2)}{40^2}(-5) + \frac{100}{40}(0.10) \\ &= 1 + \frac{5}{8} + \frac{1}{4} = \frac{15}{8} \text{ dollars}\end{aligned}$$

■

3. [2350/121125 (20 pts)] The intensity of sweet-smelling nectar is given by $N(x,y) = x^3 - 3x + y^3 - 12y + 20$. As our beloved hummingbird from this semester flies around, you need to decide if she will experience the following. Fully justify your answers.

   (a) Are there any points where the intensity will be a relative or local maximum? If so, find those points and the intensity there.

   (b) Are there any points where the intensity will increase in all directions from that point? If so, find them. If not, explain why not.

   (c) Are there points where the intensity will increase from the point in some directions and decrease in others? If so, find them. If not, explain why not.

   SOLUTION:

   This is an optimization problem.

$$N_x = 3x^2 - 3 = 0 \implies x = \pm 1$$

$$N_y = 3y^2 - 12 = 0 \implies y = \pm 2$$

$$N_{xx} = 6x \quad N_{yy} = 6y \quad N_{xy} = 0 \quad D(x,y) = 36xy$$

critical point	classification	value of N
$(1,2)$	$D(1,2) > 0, N_{xx}(1,2) > 0 \implies$ local minimum	2
$(-1,-2)$	$D(-1,-2) > 0, N_{xx}(-1,-2) < 0 \implies$ local maximum	38
$(-1,2)$	$D(-1,2) < 0 \implies$ saddle	6
$(1,-2)$	$D(1,-2) < 0 \implies$ saddle	34

   (a) Maximum intensity of 38 at $(-1,-2)$.

   (b) This refers to local or relative minima: $(1,2)$.

   (c) These are saddle points: $(-1,2), (1,-2)$.

■

4. [2350/121125 (21 pts)] Consider the vector force field $\mathbf{F} = \langle yz - \sin x \cos y, e^z - \cos x \sin y + xz, xy + ye^z \rangle$. Star Command has tasked Buzz Lightyear with going to infinity and beyond, having him compute the work required to traverse every possible closed path in $\mathbb{R}^3$. Buzz reports that no work was done on any of those closed paths.

   (a) (2 pts) What do Buzz's wanderings and calculations tell you about $\mathbf{F}$?

   (b) (10 pts) Emperor Zurg and Buzz are engaged in combat at the point $(2\pi, \pi, 1)$. Buzz wants to move away from Zurg along the path $\mathbf{r}(t) = t\,\mathbf{i} + \left(\frac{t^2}{2\pi} - 2t + 3\pi\right)\mathbf{j} + \mathbf{k}$ to the point $(4\pi, 3\pi, 1)$. Use an important Calculus 3 theorem to determine the amount of work Buzz will do escaping Zurg.

   (c) (2 pts) Emperor Zurg claims that he can find the binormal vector, $\mathbf{B}$, to Buzz's path without doing any lengthy calculations. Prove him right by finding $\mathbf{B}$.

(d) (4 pts) Buzz's fellow Space Rangers want to know what is the torsion, τ, of Buzz's path $\mathbf{r}(t)$. To aid in the computation, they provide the formula

$$\tau = \frac{[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)}{||\mathbf{r}'(t) \times \mathbf{r}''(t)||^2}$$

What value of τ should you provide to the Space Rangers?

(e) (3 pts) Suppose Buzz chose to escape Zurg along the path $\mathbf{r}_1(t) = \langle 2\pi(1+t), \pi(1+2t), 1 \rangle$, $0 \le t \le 1$ instead of the original path. Without doing any calculations would Buzz do the same amount of, less or more work following this new path? Provide a brief justification of your answer in words only.

SOLUTION:

(a) The vector field is conservative and $\mathbf{F} = \nabla f$ for some potential function f.

(b) Find the potential function for $\mathbf{F}$ and use the Fundamental Theorem for Line Integrals.

$$\frac{\partial f}{\partial x} = yz - \sin x \cos y \implies f(x,y,z) = \int (yz - \sin x \cos y)\, \mathrm{d}x = xyz + \cos x \cos y + g(y,z)$$

$$\frac{\partial f}{\partial y} = xz - \cos x \sin y + g_y(y,z) = e^z - \cos x \sin y + xz \implies g_y(y,z) = e^z$$

$$\implies g(y,z) = \int e^z\, \mathrm{d}y = ye^z + h(z) \implies f(x,y,z) = xyz + \cos x \cos y + ye^z + h(z)$$

$$\frac{\partial f}{\partial z} = xy + ye^z + h'(z) = xy + ye^z \implies h'(z) = 0 \implies h(z) = c$$

$$f(x,y,z) = xyz + \cos x \cos y + ye^z + c$$

Given the potential function, we have

$$\begin{aligned}
\text{Work} &= \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{(2\pi,\pi,1)}^{(4\pi,3\pi,1)} \nabla f \cdot \mathrm{d}\mathbf{r} = f(4\pi, 3\pi, 1) - f(2\pi, \pi, 1) \\
&= (4\pi)(3\pi)(1) + \cos 4\pi \cos 3\pi + 3\pi e^1 - \left[(2\pi)(\pi)(1) + \cos 2\pi \cos \pi + \pi e^1\right] \\
&= 12\pi^2 + (1)(-1) + 3\pi e - 2\pi^2 - (1)(-1) - \pi e \\
&= 10\pi^2 + 2\pi e = 2\pi(5\pi + e)
\end{aligned}$$

(c) The path, and therefore $\mathbf{T}$ and $\mathbf{N}$, lie in the plane $z = 1$ so $\mathbf{B} = \pm\mathbf{k}$. Zurg's claim is correct.

(d)

$$\mathbf{r}'(t) = \left\langle 1, \frac{t}{\pi} - 2, 0 \right\rangle \implies \mathbf{r}''(t) = \left\langle 0, \frac{1}{\pi}, 0 \right\rangle \implies \mathbf{r}'''(t) = \langle 0, 0, 0 \rangle$$

No need to compute the cross product as the dot product of any vector with the zero vector is zero. The torsion $\tau = 0$.

(e) Buzz would do the same amount of work. The endpoints of this new path coincide with those of the original path and line integrals of conservative vector fields are path independent.

■

5. [2350/121125 (20 pts)] Consider the surface, $\mathcal{S}$, parameterized as

$$\mathbf{r}(u,v) = \left\langle 1 - u^2, u \cos v, u \sin v \right\rangle, \quad \mathcal{R}: 0 \le u \le 1,\ 0 \le v \le 2\pi$$

oriented with normal pointing in the positive x-direction. Let $\mathbf{F} = z\,\mathbf{i} - x\,\mathbf{j} + y\,\mathbf{k}$.

(a) (10 pts) Evaluate $\iint_{\mathcal{S}} \nabla \times \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}S$ directly without using any major Calculus 3 theorems.

(b) (10 pts) Name and use a major Calculus 3 theorem to evaluate the integral using a different type of integral.

SOLUTION:

(a)

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -2u, \cos v, \sin v\rangle \times \langle 0, -u\sin v, u\cos v\rangle = \left\langle u, 2u^2\cos v, 2u^2\sin v\right\rangle \quad \text{properly oriented}$$

$$\nabla \times \mathbf{F} = \nabla \times \langle z, -x, y\rangle = \langle 1, 1, -1\rangle$$

$$\nabla \times \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = \langle 1,1,-1\rangle \cdot \left\langle u, 2u^2\cos v, 2u^2\sin v\right\rangle = u + 2u^2\cos v - 2u^2\sin v$$

$$\iint_{\mathcal{S}} \nabla \times \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iint_{\mathcal{R}} \nabla\times\mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,\mathrm{d}A = \int_0^{2\pi}\int_0^1 \left(u + 2u^2\cos v - 2u^2\sin v\right)\mathrm{d}u\,\mathrm{d}v = \pi$$

(b) Stokes' Theorem. We need to compute $\int_{\partial\mathcal{S}} \mathbf{F}\cdot\mathrm{d}\mathbf{r}$. The boundary of $\mathcal{S}$ is the unit circle in the yz-plane, oriented counterclockwise when looking in the negative x direction.

$$\partial\mathcal{S} : \mathbf{r}(t) = \langle 0, \cos t, \sin t\rangle,\ 0 \le t \le 2\pi$$

$$\mathbf{r}'(t) = \langle 0, -\sin t, \cos t\rangle$$

$$\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = \langle \sin t, 0, \cos t\rangle \cdot \langle 0, -\sin t, \cos t\rangle = \cos^2 t$$

$$\int_{\partial\mathcal{S}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_0^{2\pi} \cos^2 t\,\mathrm{d}t = \frac{1}{2}\int_0^{2\pi}(1+\cos 2t)\,\mathrm{d}t = \pi$$

■

6. [2350/121125 (34 pts)] Consider the vector field $\mathbf{F} = z\,\mathbf{i} + z\,\mathbf{j} + z^2\,\mathbf{k}$. Let $\mathcal{E}$ be the solid region beneath the cone $z = -\sqrt{x^2+y^2}$ and above plane $z = -1$ having closed boundary $\partial\mathcal{E}$.

(a) (16 pts) Find the outward flux of $\mathbf{F}$ through $\partial\mathcal{E}$ directly. You must use parameterized surfaces. Using projections will result in zero points.

(b) (10 pts) Apply Gauss' Divergence Theorem to verify your answer to part (a). Use cylindrical coordinates.

(c) (8 pts) Now set up, but **do not evaluate**, the integral(s) in spherical coordinates that would be used to apply Gauss' Divergence Theorem to verify your answer to part (a). Simplify your integrand(s).

SOLUTION:

(a) $\partial\mathcal{E}$ is a piecewise smooth surface consisting of the cone $\mathcal{S}_c$ and the disk $\mathcal{S}_d$.

- Cone, $\mathcal{S}_c$.

$$\mathbf{r}(u,v) = \langle u\cos v, u\sin v, -u\rangle, \quad \mathcal{R} : 0 \le u \le 1,\ 0 \le v \le 2\pi$$

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \cos v, \sin v, -1\rangle \times \langle -u\sin v, u\cos v, 0\rangle = \langle u\cos v, u\sin v, u\rangle \quad \text{correct (upward) orientation}$$

$$\mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v) = \left\langle -u, -u, u^2\right\rangle \cdot \langle u\cos v, u\sin v, u\rangle = -u^2\cos v - u^2\sin v + u^3$$

$$\text{Flux} = \iint_{\mathcal{S}_c} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iint_{\mathcal{R}} \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,\mathrm{d}A = \int_0^{2\pi}\int_0^1 \left(-u^2\cos v - u^2\sin v + u^3\right)\mathrm{d}u\,\mathrm{d}v$$

$$= -\left(\int_0^{2\pi}\cos v\,\mathrm{d}v\right)\left(\int_0^1 u^2\,\mathrm{d}u\right) - \left(\int_0^{2\pi}\sin v\,\mathrm{d}v\right)\left(\int_0^1 u^2\,\mathrm{d}u\right) + \int_0^{2\pi}\int_0^1 u^3\,\mathrm{d}u\,\mathrm{d}v = \frac{\pi}{2}$$

Alternatively,

$$\mathbf{r}(u,v) = \left\langle u, v, -\sqrt{u^2+v^2}\right\rangle, \quad \mathcal{R} : u^2+v^2 \le 1$$

$$\mathbf{r}_u\times\mathbf{r}_v = \left\langle 1, 0, -\frac{u}{\sqrt{u^2+v^2}}\right\rangle \times \left\langle 0, 1, -\frac{v}{\sqrt{u^2+v^2}}\right\rangle = \left\langle \frac{u}{\sqrt{u^2+v^2}}, \frac{v}{\sqrt{u^2+v^2}}, 1\right\rangle \quad \text{correct (upward) orientation}$$

$$\mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v) = \left\langle -\sqrt{u^2+v^2}, -\sqrt{u^2+v^2}, u^2+v^2\right\rangle \cdot \left\langle \frac{u}{\sqrt{u^2+v^2}}, \frac{v}{\sqrt{u^2+v^2}}, 1\right\rangle = -u - v + u^2 + v^2$$

$$\text{Flux} = \iint_{\mathcal{S}_c}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iint_{\mathcal{R}}\mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,\mathrm{d}A = \iint\limits_{u^2+v^2\le 1}\left(-u - v + u^2 + v^2\right)\mathrm{d}A \quad \text{polar coordinates}$$

$$= \int_0^{2\pi}\int_0^1 \left(-r\cos\theta - r\sin\theta + r^2\right) r\,\mathrm{d}r\,\mathrm{d}\theta \quad \text{same integral as above}$$

- Disk, $\mathcal{S}_d$.

$$\mathbf{r}(u,v) = \langle u\cos v, u\sin v, -1\rangle\,, \quad \mathcal{R}: 0 \le u \le 1,\ 0 \le v \le 2\pi$$

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \cos v, \sin v, 0\rangle \times \langle -u\sin v, u\cos v, 0\rangle = \langle 0,0,u\rangle \quad \text{wrong orientation}$$

$$\mathbf{F}(\mathbf{r}(u,v)) \cdot (-\mathbf{r}_u \times \mathbf{r}_v) = \langle -1,-1,1\rangle \cdot \langle 0,0,-u\rangle = -u$$

$$\text{Flux} = \iint_{\mathcal{S}_d} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iint_{\mathcal{R}} \mathbf{F}(\mathbf{r}(u,v))\cdot(-\mathbf{r}_u \times \mathbf{r}_v)\,\mathrm{d}A = \int_0^{2\pi}\int_0^1 -u\,\mathrm{d}u\,\mathrm{d}v = -\pi$$

Alternatively,

$$\mathbf{r}(u,v) = \langle u, v, -1\rangle\,, \quad \mathcal{R}: u^2+v^2 \le 1$$

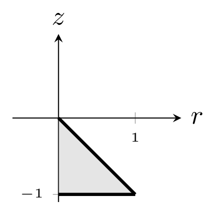
$$\mathbf{r}_u \times \mathbf{r}_v = \langle 1,0,0\rangle \times \langle 0,1,0\rangle = \langle 0,0,1\rangle \quad \text{wrong orientation}$$

$$\mathbf{F}(\mathbf{r}(u,v)) \cdot (-\mathbf{r}_u \times \mathbf{r}_v) = \langle -1,-1,1\rangle \cdot \langle 0,0,-1\rangle = -1$$

$$\text{Flux} = \iint_{\mathcal{S}_d} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iint_{\mathcal{R}} \mathbf{F}(\mathrm{r}(u,v))\cdot(-\mathrm{r}_u \times \mathbf{r}_v)\,\mathrm{d}A = \iint\limits_{u^2+v^2\le 1} -1\,\mathrm{d}A = -\,\text{area}\left(u^2+v^2 \le 1\right) = -\pi$$

Therefore the flux through $\partial\mathcal{E}$ = flux through cone + flux through disk $= \pi/2 + (-\pi) = -\pi/2$.

(b) $\nabla\cdot\mathbf{F} = 2z$. Here is a sketch of the region in a constant θ plane.



$$\text{Flux} = \int_0^{2\pi}\int_0^1\int_{-1}^{-r} 2zr\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta = 2\int_0^{2\pi}\int_0^1 \frac{z^2}{2}\Big|_{-1}^{-r} r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\int_0^1 \left(r^3 - r\right)\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\left(\frac{r^4}{4} - \frac{r^2}{2}\right)\Big|_0^1 \mathrm{d}\theta = -\frac{\pi}{2}$$

Alternatively,

$$\text{Flux} = \int_0^{2\pi}\int_{-1}^0\int_0^{-z} 2zr\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta = 2\int_0^{2\pi}\int_{-1}^0 \frac{r^2}{2}\Big|_0^{-z} z\,\mathrm{d}z\,\mathrm{d}\theta = \int_0^{2\pi}\int_{-1}^0 z^3\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi} \frac{z^4}{4}\Big|_{-1}^0 \mathrm{d}\theta = -\frac{\pi}{2}$$

(c)

$$\int_0^{2\pi}\int_{3\pi/4}^{\pi}\int_0^{-\sec\phi} 2\rho^3\cos\phi\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

7. [2350/121125 (23 pts)] Let $\mathcal{D}$ be the triangle in the xy-plane with vertices $(0,0)$, $(0,-2)$, $(1,0)$. Let $\partial\mathcal{D}$ be the boundary of $\mathcal{D}$ oriented counterclockwise. Consider the vector field $\mathbf{F} = \left\langle xy^2 + x, xy\right\rangle$.

(a) (10 pts) Evaluate $\displaystyle\int_{\partial\mathcal{D}} \left(xy^2+x\right)\mathrm{d}y - xy\,\mathrm{d}x$ directly, that is, without using any major Calculus 3 theorems.

(b) (3 pts) What quantity does the integral in part (a) represent?

(c) (10 pts) Use Green's Theorem to set up, **but not evaluate**, two equivalent integrals to compute the circulation of $\mathbf{F}$ on $\partial\mathcal{D}$: one using the order $\mathrm{d}y\,\mathrm{d}x$ and the second one using the order $\mathrm{d}x\,\mathrm{d}y$.

SOLUTION:

(a) The boundary, $\partial\mathcal{D}$, is piecewise smooth, consisting of three smooth pieces: one along the x-axis ($\mathcal{C}_1$), another along the y-axis ($\mathcal{C}_2$) and a third along the line $y = 2x - 2$ ($\mathcal{C}_3$). Thus

$$\int_{\partial\mathcal{D}} \left(xy^2+x\right)\mathrm{d}y - xy\,\mathrm{d}x = \int_{\mathcal{C}_1} \left(xy^2+x\right)\mathrm{d}y - xy\,\mathrm{d}x + \int_{\mathcal{C}_2} \left(xy^2+x\right)\mathrm{d}y - xy\,\mathrm{d}x + \int_{\mathcal{C}_3} \left(xy^2+x\right)\mathrm{d}y - xy\,\mathrm{d}x$$

$$\mathcal{C}_1 : y = 0, \mathrm{d}y = 0\,\mathrm{d}t \implies \int_{\mathcal{C}_1} \left(xy^2 + x\right) \mathrm{d}y - xy\,\mathrm{d}x = 0$$

$$\mathcal{C}_2 : x = 0, \mathrm{d}x = 0\,\mathrm{d}t \implies \int_{\mathcal{C}_2} \left(xy^2 + x\right) \mathrm{d}y - xy\,\mathrm{d}x = 0$$

$$\mathcal{C}_3 : x = t, y = 2(t-1), 0 \le t \le 1, \mathrm{d}x = \mathrm{d}t, \mathrm{d}y = 2\,\mathrm{d}t$$

$$\implies \int_{\mathcal{C}_3} \left(xy^2 + x\right) \mathrm{d}y - xy\,\mathrm{d}x = \int_0^1 \left\{t[2(t-1)]^2 + t\right\}(2\,\mathrm{d}t) - t(2)(t-1)\,\mathrm{d}t$$

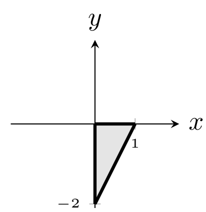
$$= 2\int_0^1 \left[4t\left(t^2 - 2t + 1\right) + t - t^2 + t\right] \mathrm{d}t = 2\int_0^1 \left(4t^3 - 9t^2 + 6t\right) \mathrm{d}t = 2\left(t^4 - 3t^3 + 3t^2\right)\Big|_0^1 = 2$$

Therefore,

$$\int_{\partial\mathcal{D}} \left(xy^2 + x\right) \mathrm{d}y - xy\,\mathrm{d}x = 2$$

(b) It represents the flux of $\mathbf{F}$ through or across $\partial\mathcal{D}$.

(c) The circulation is $\int_{\partial\mathcal{D}} \left(xy^2 + x\right) \mathrm{d}x + xy\,\mathrm{d}y = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A$ from Green's Theorem. The region, $\mathcal{D}$, is the interior of the triangle:



$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}\left(xy^2 + x\right) = y - 2xy = y(1-2x)$$

$$\text{Circulation} = \int_0^1 \int_{2x-2}^0 y\,(1-2x)\,\mathrm{d}y\,\mathrm{d}x = \int_{-2}^0 \int_0^{1+y/2} y\,(1-2x)\,\mathrm{d}x\,\mathrm{d}y$$