- This exam is worth 150 points and has 7 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"$\times$ 11" crib sheet with writing on two sides.

0. At the top of the page containing your solution to problem 1, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2350/121125 (24 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a two-column table (letter - answer) completely separate from any work you do to arrive at the answer.

   (a) $\mathbf{r}(u,v) = \langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle, 0 \le u \le \pi, 0 \le v \le \pi/2$ parameterizes the first octant portion of the sphere of radius $a > 0$ with center at $(0,0,0)$.

   (b) The first order Taylor polynomial of $f(x,y) = e^{-x^2-y^2}$ centered at the origin is the plane $T_1(x,y) = 1$.

   (c) If $u = xy$ and $v = xy^2$, the Jacobian used in the change of variables theorem for double integrals is v.

   (d) There exists a point in $\mathbb{R}^3$ where the plane $x + y + z = 1$ and the line with symmetric equations $\dfrac{x}{2} = -y = -z$ intersect.

   (e) $f(x,y) = \dfrac{xy}{x^2+y^2}$ is continuous on its domain.

   (f) Suppose you are traveling along the differentiable path $\mathbf{r}(s)$ in $\mathbb{R}^2$, where s is the arc length parameter. At the point (a,b), which is not a critical point of the differentiable function $f(x,y)$, the directional derivative equals 0. Then you are instantaneously traveling in the direction of the gradient vector.

   (g) For c a real number, $cx^2 + y^2 + z^2 = 1$ describes either an ellipsoid, a sphere or hyperboloid of one sheet.

   (h) Let $f(x,y) = x^2 + xy + y^2$. For all (x,y) in $\mathbb{R}^2$, the rate of change in the y-direction of the slope of the tangent line in the x-direction is 1.

2. [2350/121125 (8 pts)] The cost, C, of a trip that is L miles long, driving a car that gets m miles per gallon, with gas costs of p dollars per gallon is $C = Lp/m$ dollars. You are planning a trip of $L = 100$ miles in a car whose gas mileage is $m = 40$ miles per gallon with the price of gas at $p = \$2.00$ per gallon. Use differentials to estimate the change in the trip's cost if you get detoured an extra 20 miles which lowers your gas mileage by 5 miles per gallon and requires you to pay \$0.10 more per gallon of gas.

3. [2350/121125 (20 pts)] The intensity of sweet-smelling nectar is given by $N(x,y) = x^3 - 3x + y^3 - 12y + 20$. As our beloved hummingbird from this semester flies around, you need to decide if she will experience the following. Fully justify your answers.

   (a) Are there any points where the intensity will be a relative or local maximum? If so, find those points and the intensity there.

   (b) Are there any points where the intensity will increase in all directions from that point? If so, find them. If not, explain why not.

   (c) Are there points where the intensity will increase from the point in some directions and decrease in others? If so, find them. If not, explain why not.

MORE PROBLEMS BELOW/ON REVERSE

4. [2350/121125 (21 pts)] Consider the vector force field $\mathbf{F} = \langle yz - \sin x \cos y, e^z - \cos x \sin y + xz, xy + ye^z \rangle$. Star Command has tasked Buzz Lightyear with going to infinity and beyond, having him compute the work required to traverse every possible closed path in $\mathbb{R}^3$. Buzz reports that no work was done on any of those closed paths.

   (a) (2 pts) What do Buzz's wanderings and calculations tell you about $\mathbf{F}$?

   (b) (10 pts) Emperor Zurg and Buzz are engaged in combat at the point $(2\pi, \pi, 1)$. Buzz wants to move away from Zurg along the path $\mathbf{r}(t) = t\,\mathbf{i} + \left(\frac{t^2}{2\pi} - 2t + 3\pi\right)\mathbf{j} + \mathbf{k}$ to the point $(4\pi, 3\pi, 1)$. Use an important Calculus 3 theorem to determine the amount of work Buzz will do escaping Zurg.

   (c) (2 pts) Emperor Zurg claims that he can find the binormal vector, $\mathbf{B}$, to Buzz's path without doing any lengthy calculations. Prove him right by finding $\mathbf{B}$.

   (d) (4 pts) Buzz's fellow Space Rangers want to know what is the torsion, τ, of Buzz's path $\mathbf{r}(t)$. To aid in the computation, they provide the formula

$$\tau = \frac{[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)}{||\mathbf{r}'(t) \times \mathbf{r}''(t)||^2}$$

   What value of τ should you provide to the Space Rangers?

   (e) (3 pts) Suppose Buzz chose to escape Zurg along the path $\mathbf{r}_1(t) = \langle 2\pi(1+t), \pi(1+2t), 1 \rangle$, $0 \le t \le 1$ instead of the original path. Without doing any calculations would Buzz do the same amount of, less or more work following this new path? Provide a brief justification of your answer in words only.

5. [2350/121125 (20 pts)] Consider the surface, $\mathcal{S}$, parameterized as

$$\mathbf{r}(u,v) = \left\langle 1 - u^2, u\cos v, u \sin v \right\rangle, \quad \mathcal{R}: 0 \le u \le 1,\ 0 \le v \le 2\pi$$

oriented with normal pointing in the positive x-direction. Let $\mathbf{F} = z\,\mathbf{i} - x\,\mathbf{j} + y\,\mathbf{k}$.

   (a) (10 pts) Evaluate $\iint_{\mathcal{S}} \nabla \times \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}S$ directly without using any major Calculus 3 theorems.

   (b) (10 pts) Name and use a major Calculus 3 theorem to evaluate the integral using a different type of integral.

6. [2350/121125 (34 pts)] Consider the vector field $\mathbf{F} = z\,\mathbf{i} + z\,\mathbf{j} + z^2\,\mathbf{k}$. Let $\mathcal{E}$ be the solid region beneath the cone $z = -\sqrt{x^2+y^2}$ and above plane $z = -1$ having closed boundary $\partial\mathcal{E}$.

   (a) (16 pts) Find the outward flux of $\mathbf{F}$ through $\partial\mathcal{E}$ directly. You must use parameterized surfaces. Using projections will result in zero points.

   (b) (10 pts) Apply Gauss' Divergence Theorem to verify your answer to part (a). Use cylindrical coordinates.

   (c) (8 pts) Now set up, but **do not evaluate**, the integral(s) in spherical coordinates that would be used to apply Gauss' Divergence Theorem to verify your answer to part (a). Simplify your integrand(s).

7. [2350/121125 (23 pts)] Let $\mathcal{D}$ be the triangle in the the xy-plane with vertices $(0,0)$, $(0,-2)$, $(1,0)$. Let $\partial\mathcal{D}$ be the boundary of $\mathcal{D}$ oriented counterclockwise. Consider the vector field $\mathbf{F} = \left\langle xy^2 + x, xy \right\rangle$.

   (a) (10 pts) Evaluate $\int_{\partial\mathcal{D}} \left(xy^2 + x\right) \mathrm{d}y - xy\, \mathrm{d}x$ directly, that is, without using any major Calculus 3 theorems.

   (b) (3 pts) What quantity does the integral in part (a) represent?

   (c) (10 pts) Use Green's Theorem to set up, **but not evaluate**, two equivalent integrals to compute the circulation of $\mathbf{F}$ on $\partial\mathcal{D}$: one using the order $\mathrm{d}y\,\mathrm{d}x$ and the second one using the order $\mathrm{d}x\,\mathrm{d}y$.