APPM 3570/STAT 3100 — Exam 2 — Fall 2025

On the front of your bluebook, write (1) **your name**, (2) **Exam 2**, (3) **APPM 3570/STAT 3100**. Correct answers with no supporting work may receive little or no credit. Books, notes and electronic devices of any kind are not allowed. Your exam should be uploaded to Gradescope in a PDF format (Recommended: **Genius Scan**, **Scannable** or **CamScanner** for iOS/Android). **Show all work, justify your answers. Do all problems.** Students are required to re-write the **honor code statement** in the box below on the **first page** of their exam submission and **sign and date it**:

On my honor, as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work. Signature:____________________________ Date:_________

1. [EXAM02] (32pts) There are 4 unrelated parts to this question. Justify your answers.

   (a) (8pts) The chance that a person will independently believe a rumor about a certain politician is $\frac{3}{4}$. Find the probability that the eighth person to hear the rumor will be the fifth person to believe it.

   (b) (8pts) Find the *probability density function* of $Y = e^X$ when X is normally distributed with parameters μ and σ^2. No need to verify the pdf. (Be sure to define the pdf for all real values.)

   (c) (8pts) Consider a fair 4 sided dice (shaped like a 4 sided pyramid) labeled 1, 2, 3, 4. When it is rolled it lands face down and the number that the die lands on is considered to be the outcome of the die roll. Now suppose the tetrahedral dice is rolled twice. Let X equal 1 if the first roll is even and 0 otherwise and let Y equal 1 if the first roll minus the second roll is negative and 0 otherwise. Find the joint pmf of (X, Y). No need to verify. (Be sure to define the pmf for all real values.)

   (d) (8pts) If the joint probability mass function of the discrete random variables X, Y and Z is given by $p(x, y, z) = \dfrac{(x+y)z}{63}$ for $x = 1, 2$; $y = 1, 2, 3$; $z = 1, 2$ (and 0 otherwise). Find $P(X = 2, Y + Z \leq 3)$.

Solution:

(a)(8pts) Consider each person to be a "trial" and let the event that a person believes the rumor be considered a "success", then let X be the number of trials until the fifth success, then X is a negative binomial random variable and

$$P(X = 5) = \binom{7}{4}\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^3 \cdot \left(\frac{3}{4}\right) = \boxed{\binom{7}{4} \cdot \frac{3^5}{4^8}} \approx 0.1298.$$

(b)(8pts) Let $X \sim \text{Normal}(\mu, \sigma^2)$ and let $Y = e^X$ then note that $Y > 0$ and the cdf of Y is

$$F_Y(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq \ln(Y)) = F_X(\ln(Y))$$

now note that $F_X(x) = \Phi\left(\dfrac{x-\mu}{\sigma}\right)$ thus $F_Y(y) = F_X(\ln(y)) = \Phi\left(\dfrac{\ln(y)-\mu}{\sigma}\right)$ and so the pdf of Y is

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}\left[\Phi\left(\frac{\ln(y)-\mu}{\sigma}\right)\right] = \Phi'\left(\frac{\ln(y)-\mu}{\sigma}\right) \cdot \frac{1}{\sigma} \cdot \frac{1}{y}$$

and note that $\Phi'(x) = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$ and so we see that the pdf of the *lognormal distribution* is

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma y}e^{-(\ln(y)-\mu)^2/2\sigma^2}, \quad y > 0 \text{ and } f_Y(y) = 0 \text{ otherwise.}}$$

(c)(10pts) A table is nice:

Roll #1 one minus Roll #2

R1 \ R2	1	2	3	4
1	0	−1	−2	−3
2	1	0	−1	−2
3	2	1	0	−1
4	3	2	1	0

Since $X = \begin{cases} 1, & \text{Roll 1 is even} \\ 0, & \text{else} \end{cases}$ and $Y = \begin{cases} 1, & \text{Roll 1 minus Roll 2 is negative} \\ 0, & \text{else} \end{cases}$, the observable values of (X,Y) are $\{(0,0),(0,1),(1,0),(1,1)\}$, with joint pmf:

$$P(X=0,Y=0) = P(\{(1,1),(3,1),(3,2),(3,3)\}) = \frac{4}{16}$$
$$P(X=0,Y=1) = P(\{(1,2),(1,3),(1,4),(3,4)\}) = \frac{4}{16}$$
$$P(X=1,Y=0) = P(\{(2,1),(2,2),(4,1),(4,2),(4,3),(4,4)\}) = \frac{6}{16}$$
$P(X=1,Y=1) = P(\{(2,3),(2,4)\}) = \frac{2}{16}$ and $P(X=i,Y=j)=0$ otherwise.

(d)(8pts) Note that $\{Y+Z \le 3\} = \{(Y=1,Z=1),(Y=1,Z=2),(Y=2,Z=1)\}$ so

$$P(X=2,Y+Z\le 3) = p(2,1,1)+p(2,1,2)+p(2,2,1) = \frac{(2+1)1}{63}+\frac{(2+1)2}{63}+\frac{(2+2)1}{63} = \frac{3+6+4}{63} = \boxed{\frac{13}{63}}$$

2. [Exam02] (40pts) The weekly demand for propane gas (in 1000s of gallons) from a particular facility is a random variable X with probability density function

$$f(x) = \begin{cases} 2 - \dfrac{2}{x^2}, & \text{if } 1 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

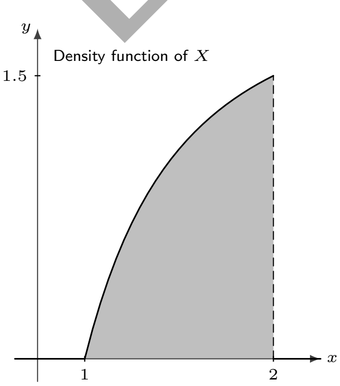
(a) (10pts) What is the probability that the weekly demand for propane gas from this facility will be at most 1,500 gallons? *Simplify your answer.*

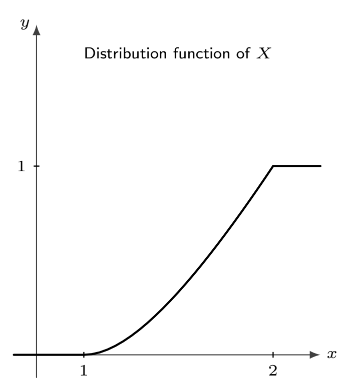
(b) (10pts) Find the **cumulative distribution function** of the random variable X. (Be sure to define the cdf for all real values.)

(c) (10pts) Find the expectation of X.

(d) (10pts) Find the variance of X.

Solution:





(a)(10pts) The weekly demand for propane gas from this facility will be at most 1,500 gallons if and only if $\{X \leq 1.5\}$, thus,

$$P(X \leq 1.5) = P(1 < X \leq 1.5) = \int_1^{1.5} \Big(2 - \frac{2}{x^2}\Big)\,dx = \Big(2x + \frac{2}{x}\Big)\Big|_1^{3/2} = \Big(3 + \frac{4}{3}\Big) - 4 = \boxed{\frac{1}{3}}$$

(b)(10pts) Note that for $a \in (1, 2)$, we have

$$F_X(a) = P(X \leq a) = \int_1^a \Big(2 - \frac{2}{x^2}\Big)\,dx = \Big(2x + \frac{2}{x}\Big)\Big|_1^a = \Big(2a + \frac{2}{a}\Big) - 4$$

thus the cdf of X is

$$\boxed{F_X(x) = \begin{cases} 0, & \text{if } x < 1, \\ 2x + \frac{2}{x} - 4, & \text{if } 1 \leq x < 2, \\ 1, & \text{if } x \geq 2. \end{cases}}$$

(c)(10pts) Note that the expectation of X is

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_1^2 \Big(2x - \frac{2}{x}\Big)\,dx = \Big(x^2 - 2\ln(x)\Big)\Big|_1^2 = \boxed{3 - 2\ln(2)} \approx 1.6137.$$

(d)(10pts) Note that

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_1^2 \Big(2x^2 - 2\Big)\,dx = \Big(2\frac{x^3}{3} - 2x\Big)\Big|_1^2 = \Big(2 \cdot \frac{8}{3} - 4\Big) - \Big(2 \cdot \frac{1}{3} - 2\Big) = \frac{14}{3} - 2 = \frac{8}{3}.$$

Thus, the variance of X is

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = \boxed{\frac{8}{3} - \big[3 - 2\ln(2)\big]^2} \approx 0.0626.$$

3. [EXAM02] (28pts) If the joint probability density function of X and Y is given by

$$f(x, y) = \begin{cases} \dfrac{1}{y}, & \text{for } 0 < x < y,\ 0 < y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

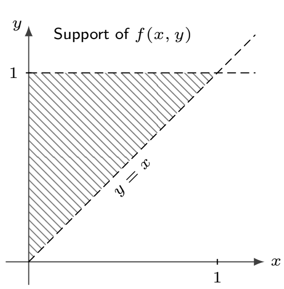
(a) (7pts) Set-up, but *do not solve* an integral (or integrals) to find the probability that the sum of the values of X and Y will not exceed $\frac{1}{2}$.

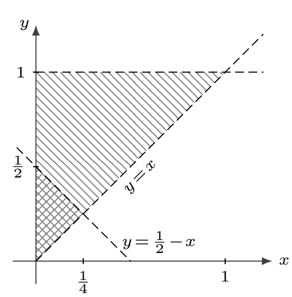
(b) (7pts) Find $f_X(x)$, the marginal pdf of X, and then *verify* that it satisfies the definition of a pdf. (Be sure to define the pdf for all real values.)

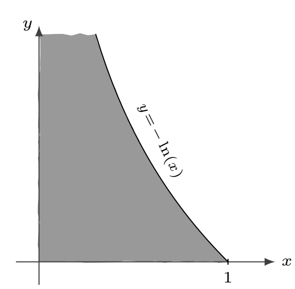
(c) (7pts) Find $\mathrm{Var}(2Y + 3)$. Justify your answer.

(d) (7pts) Is it true that $f(x, y) = f_X(x) f_Y(y)$? What can you conclude from this? (Be sure to explicitly answer both questions.)

Solution:







(a)(7pts) Note that

$$P(X+Y\le \tfrac{1}{2}) = P(Y\le \tfrac{1}{2}-X) = P(0<X<\tfrac{1}{4}, X<Y<\tfrac{1}{2}-X) = \boxed{\int_0^{\frac{1}{4}}\int_x^{\frac{1}{2}-x}\frac{1}{y}\,dydx}$$

(b)(7pts) For each $x\in(0,1)$, we have

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^1 \frac{1}{y}\,dy = \ln(y)\Big|_x^1 = -\ln(x) \quad\Rightarrow\quad \boxed{f_X(x) = -\ln(x) \text{ for } 0<x<1 \text{ and } 0 \text{ else.}}$$

Verify Note that, using integration by parts, with $u=-\ln(x)\Rightarrow du=-\frac{1}{x}$ and $dv=dx\Rightarrow v=x$, we have

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 -\ln(x)\,dx = -x\ln(x)\Big|_0^1 + \int_0^1 \frac{1}{x}\cdot x\,dx = \underbrace{\lim_{x\to 0^+} -x\ln(x)}_{=0 \text{ by L'H}} + 1 = 1$$

where $\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+}\frac{\ln(x)}{1/x} \stackrel{L'H}{=} \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0$ and $f(x)=-\ln(x)>0$ for $0<x<1$ and 0 otherwise. Therefore, $f(x)$ is a pdf.

(c)(7pts) Note the marginal of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^y \frac{1}{y}\,dx = \frac{1}{y}x\Big|_0^y = \frac{y}{y} = 1 \text{ for } 0<y<1 \text{ and } 0 \text{ else} \quad\Rightarrow\quad Y\sim\text{Uniform}(0,1).$$

Since $Y\sim\text{Uniform}(0,1)$, we know from a result in class that $\text{Var}(Y)=\frac{1}{12}$ thus

$$\text{Var}(2Y+3) = 4\text{Var}(Y) = 4\cdot\frac{1}{12} = \boxed{\frac{1}{3}}$$

(d)(7pts) No. Clearly $f(x,y)=\frac{1}{y}\neq -\ln(x)\cdot 1 = f_X(x)f_Y(y)$. We can conclude that X and Y are **not independent** random variables.