1. (30 points, 10 points each)

   Identify each of the following series as absolutely convergent, conditionally convergent, or divergent. Justify your work. As with all problems on this exam, name any test or theorem that you use.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n^2)}$

   (b) $\displaystyle\sum_{j=1}^{\infty} \frac{(3j)^j}{e^j(j+1)^j}$

   (c) $\displaystyle\sum_{k=1}^{\infty} \frac{k+5}{(k^7+k^2)^{1/3}}$

Solution:

(a) Divergent by the Integral Test. Let $f(x) = \dfrac{1}{x\ln(x^2)} = \dfrac{1}{2x\ln x}$, a positive, continuous, decreasing function for $x \geq 2$.

$$\int_2^\infty \underbrace{\frac{dx}{2x\ln x}}_{\substack{u=\ln x\\ du=dx/x}} = \int_{\ln 2}^\infty \frac{du}{2u} = \lim_{t\to\infty}\int_{\ln 2}^t \frac{du}{2u} = \lim_{t\to\infty}\left[\frac{1}{2}\ln|u|\right]_{\ln 2}^t = \lim_{t\to\infty}\frac{1}{2}\left(\ln t - \ln(\ln 2)\right) = \infty.$$

Since the integral diverges, the given series also diverges.

(b) Divergent by Root Test,

$$\lim_{n\to\infty}\left(|a_n|\right)^{1/n} = \lim_{n\to\infty}\left(\frac{(3n)^n}{e^n(n+1)^n}\right)^{1/n} = \lim_{n\to\infty}\frac{3n}{e(n+1)} = \frac{3}{e} > 1$$

so the series is divergent by the Root Test.

Alternate Solution.

Divergent by the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3(n+1))^{n+1}}{e^{n+1}(n+2)^{n+1}}\cdot\frac{e^n(n+1)^n}{(3n)^n}\right| = \lim_{n\to\infty}\left|\frac{1}{e}\cdot\frac{3(n+1)}{n+2}\left(\frac{3(n+1)^2}{3n(n+2)}\right)^n\right|$$
$$\overset{DOP}{=} \lim_{n\to\infty}\left|\frac{3}{e}\cdot\left(1+\frac{1}{n(n+2)}\right)^n\right| \overset{L'H}{=} \frac{3}{e} > 1$$

so the series is divergent by the Ratio Test.

(c) Note that these terms are positive, so if it converges it will be absolutely convergent. We apply the limit comparison test with $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^{4/3}}$:

$$\lim_{k\to\infty}\frac{\left(\frac{k+5}{(k^7+k^2)^{1/3}}\right)}{1/k^{4/3}} = \lim_{k\to\infty}\frac{k\left(1+\frac{5}{k}\right)}{k^{7/3}\left(1+\frac{1}{k^5}\right)^{1/3}}\cdot k^{4/3}$$
$$= \lim_{k\to\infty}\frac{1+\frac{5}{k}}{\left(1+\frac{1}{k^5}\right)^{1/3}}$$
$$= 1.$$

Hence, this series is absolutely convergent by the limit comparison test, since $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{4/3}}$ is a convergent p-series.

Alternate Solution.

Use direct comparison test.

$$\frac{k+5}{(k^7+k^2)^{1/3}} \le \frac{k+5}{(k^7)^{1/3}} = \frac{k}{k^{7/3}} + \frac{5}{k^{7/3}} = \frac{1}{k^{4/3}} + \frac{5}{k^{7/3}}$$

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{4/3}}$ and $\displaystyle\sum_{k=1}^{\infty} \frac{5}{k^{7/3}}$ are each convergent p-series with $p = 4/3$ and $p = 7/3$, respectively. Hence, the original series converges by the direct comparison test.

2. (20 points) Consider the power series $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{2^n n}$.

   (a) Determine the radius of convergence for $f(x)$.

   (b) Determine the interval of convergence for $f(x)$.

   (c) For the following questions, you may use inequalities, interval notation, or list all the values. You may also say "there are no such x-values" if there are none.

      i. For which x values is this series absolutely convergent?

      ii. For which x-values is this series conditionally convergent?

      iii. For which x-values is this series divergent?

Solution:

(a) Using the *Root Test*, we have that

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left|\frac{(3x-2)^n}{2^n n}\right|^{1/n} = \lim_{n\to\infty} \frac{|3x-2|}{2\sqrt[n]{n}} = \frac{|3x-2|}{2}.$$

Alternatively: Using the *Ratio Test*, we have that

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(3x-2)^{n+1}}{2^{n+1}(n+1)} \frac{2^n n}{(3x-2)^n}\right| = \lim_{n\to\infty} \frac{n}{n+1}\frac{|3x-2|}{2} = \frac{|3x-2|}{2}.$$

Regardless of the test used, for absolute convergence, we need $L < 1$ which implies

$$\frac{|3x-2|}{2} < 1 \implies \left|x - \frac{2}{3}\right| < \frac{2}{3}$$

which tells us that the radius is $\boxed{R = \frac{2}{3}.}$

(b) To find the interval of convergence, we can break our inequality from part a into

$$-\frac{2}{3} < x - \frac{2}{3} < \frac{2}{3} \implies 0 < x < \frac{4}{3}.$$

Checking the left endpoint $x = 0$, we have that

$$\sum_{n=1}^{\infty} \frac{(3\cdot 0-2)^n}{2^n n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

which is the conditionally convergent alternating harmonic series. The right endpoint $x = \frac{4}{3}$ yields the series

$$\sum_{n=1}^{\infty} \frac{\left(3 \cdot \frac{4}{3} - 2\right)^n}{2^n n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which is the divergent harmonic series. Putting everything together, the interval of convergence is

$$\boxed{I = \left[0, \frac{4}{3}\right).}$$

(c) From the previous parts, we have that

i. The series is absolutely convergent for $\boxed{(0, 4/3).}$

ii. The series is conditionally convergent for $\boxed{x = 0}$

iii. The series is divergent for $\boxed{(-\infty, 0) \cup [4/3, \infty).}$

3. (24 points) For parts (a) and (b) below, your answer should be written using sigma notation.

(a) Find the Maclaurin series for $f(x) = \dfrac{\cos x - 1}{x}$. (Use $f(0) = 0$.)

(b) Evaluate the indefinite integral $\displaystyle\int \frac{\cos x - 1}{x}\, dx$ as an infinite series.

(c) Estimate the error in evaluating $\displaystyle\int_0^1 \frac{\cos x - 1}{x}\, dx$ using the first three nonzero terms of the series you found in part (a). Leave your answer in factored form.

Solution:

(a) The Maclaurin series for $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$ meaning

$$f(x) = \frac{\cos x - 1}{x} = \frac{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) - 1}{x} = -\frac{x}{2!} + \frac{x^3}{4!} + \cdots = \boxed{\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n)!}.}$$

(b) Integrating our series from part a yields

$$\int \frac{\cos x - 1}{x}\, dx = \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n)!} \int x^{2n-1}\, dx = \boxed{C + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!2n}.}$$

(c) Plugging in bounds into our series from part b gives

$$\int_0^1 \frac{\cos x - 1}{x}\, dx = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!2n}\Bigg|_0^1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!2n}.$$

This series is alternating with $b_n = \frac{1}{(2n)!2n}$. Since $b_n \to 0$ as $n \to \infty$ and $b_{n+1} < b_n$, the *Alternating Series Estimation Theorem* applies. If we use the first three nonzero terms to approximate the integral, the error will be bounded by the fourth term, b_4. In other words, the error is bounded by

$$\boxed{|\text{error}| \le b_4 = \frac{1}{(2 \cdot 4)! \cdot 2 \cdot 4} = \frac{1}{8! \cdot 8} = \frac{1}{322560}.}$$

4. The following questions are unrelated.

(a) (10 points) Let $f(x) = (1+x)^{1/2}$. Find the second-order Taylor polynomial, $T_2(x)$, for f **centered at** $\mathbf{a = 3}$.

(b) (6 points) Find the value of $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n}}{4^{2n}(2n)!}$

(c) (10 points) Find the value of $\displaystyle\sum_{k=1}^{\infty}\frac{2}{k(k+2)}$. (Hint: use partial fractions to first find a simple expression for s_n, the n^{th} partial sum of the series.)

Solution:

(a) Computing derivatives of $f(x)$ leads to

$$\begin{aligned} f(x) &= (1+x)^{1/2} & f(3) &= 2 \\ f'(x) &= \frac{1}{2}(1+x)^{-1/2} & f'(3) &= \frac{1}{4} \\ f''(x) &= -\frac{1}{4}(1+x)^{-3/2} & f''(3) &= -\frac{1}{32}. \end{aligned}$$

Then, $T_2(x)$ is given by

$$T_2(x) = f(3) + \frac{f'(3)}{1!}(x-3) + \frac{f''(3)}{2!}(x-3)^2 = \boxed{2 + \frac{1}{4}(x-3) - \frac{1}{64}(x-3)^2.}$$

(b) Recall, the Maclaurin series for $\cos x$ is given by

$$\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}.$$

Then, manipulating the series given, we find

$$\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n}}{4^{2n}(2n)!} = \sum_{n=0}^{\infty}(-1)^n \frac{(\pi/4)^{2n}}{(2n)!} = \cos\frac{\pi}{4} = \boxed{\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.}$$

(c) From partial fractions, we find that

$$\sum_{k=1}^{\infty}\frac{2}{k(k+2)} = \sum_{k=1}^{\infty}\frac{1}{k} - \frac{1}{k+2}$$

which looks like a telescoping series. To find the sum, we can first look at the partial sums

$$\begin{aligned} s_n = \sum_{k=1}^{n}\frac{1}{k} - \frac{1}{k+2} &= \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \cancel{\cdots} + \left(\cancel{\frac{1}{k-1}} - \frac{1}{k+1}\right) + \left(\cancel{\frac{1}{k}} - \frac{1}{k+2}\right) \\ &= \frac{1}{1} + \frac{1}{2} - \frac{1}{k+1} - \frac{1}{k+2}. \end{aligned}$$

Then, the sum is given by the limit of the partial sums

$$\sum_{k=1}^{\infty}\frac{2}{k(k+2)} = \lim_{k\to\infty} s_n = \lim_{k\to\infty}\frac{1}{1} + \frac{1}{2} - \frac{1}{k+1} - \frac{1}{k+2} = \frac{1}{1} + \frac{1}{2} = \boxed{\frac{3}{2}.}$$