1. [2360/111225 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

   (a) The equation of motion for any damped harmonic oscillator, forced or not, will always have a transient part.

   (b) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\} = e^{-t}\cos t$

   (c) The equation of motion for an undamped, unforced oscillator having a mass of 16 kg, restoring (spring) constant of 4 N/m, initially displaced 1 m to the left of the equilibrium position and given a velocity of 0.5 m/s to the left can be written as either $x(t) = -\cos\dfrac{t}{2} - \sin\dfrac{t}{2}$ or $x(t) = \sqrt{2}\cos\left(\dfrac{t}{2} - \dfrac{\pi}{4}\right)$.

   (d) The Euler-Cauchy differential equation $at^2y'' + bty' + cy = 0,\ t > 0$ where a, b, c are real numbers, can be written in the form $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\vec{\mathbf{x}}$ is a column vector and $\mathbf{A}$ is a 2×2 matrix.

   (e) The potential energy of the conservative system $2\ddot{y} + ye^y = 0$ is $e^y(y-1)$.

SOLUTION:

(a) **TRUE** The roots of the characteristic equation of $m\ddot{x} + b\dot{x} + kx = 0$ are either negative real numbers or complex numbers having negative real parts. Thus the solutions will always contain an exponential function having a negative argument which approaches 0 as $t \to \infty$.

(b) **FALSE**

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{(s+1)^2+1}\right\} = \mathscr{L}^{-1}\left\{\frac{s+1-1}{(s+1)^2+1}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} + \mathscr{L}^{-1}\left\{\frac{-1}{(s+1)^2+1}\right\} = e^{-t}\left(\cos t - \sin t\right)$$

(c) **FALSE** $x(t) = -\cos\dfrac{t}{2} - \sin\dfrac{t}{2} = \sqrt{2}\cos\left(\dfrac{t}{2} - \dfrac{5\pi}{4}\right)$

(d) **TRUE** The equation is linear. Let $x_1 = y, x_2 = y'$ and $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then $x_1' = x_2$ and

$$x_2' = y'' = -\frac{c}{at^2}y - \frac{bt}{at^2}y' = -\frac{c}{at^2}x_1 - \frac{b}{at}x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -c/at^2 & -b/at \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies \vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$$

(e) **TRUE** The equation is in the form $m\ddot{y} + V'(y) = 0$ so the potential energy is $\displaystyle\int ye^y\,\mathrm{d}y = e^y(y-1)$

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2. [2360/111225 (26 pts)] An harmonic oscillator consists of a 2 kg mass attached to a spring with restoring (spring) constant of 2 N/m. The damping force is numerically equal to four times the instantaneous velocity. The oscillator is subject to an external force equal to $12te^{-t}$. Initially, the mass is located 9 m to the right of the equilibrium position and provided with a velocity of 19 m/s to the left.

   (a) (3 pts) Is the oscillator overdamped, underdamped, undamped or critically damped? Justify your answer.

   (b) (20 pts) Use variation of parameters to find the equation of motion of the oscillator. No points will be awarded for using any other technique.

   (c) (3 pts) Is the mass at the equilibrium position when $t = 1$? Justify your answer.

SOLUTION:

With $x(t)$ representing the displacement of the mass from its equilibrium position, the initial value problem is

$$2\ddot{x} + 4\dot{x} + 2x = 12te^{-t},\ x(0) = 9,\ \dot{x}(0) = -19$$

(a) Since $m = 2, b = 4$ and $k = 2$, we have $b^2 - 4mk = 16 - 4(2)(2) = 0$ and the oscillator is critically damped.

(b) The characteristic equation of the associated unforced oscillator is $2r^2 + 4r + 2 = 2(r^2 + 2r + 1) = (r + 1)^2 = 0$ implying that $r = -1$ with multiplicity of 2. A basis for the solution space of the homogeneous equation is $\{e^{-t}, te^{-t}\}$. The differential equation in standard form is $\ddot{x} + 2\dot{x} + x = 6te^{-t}$. For variation of parameters, $y_1 = e^{-t}, y_2 = te^{-t}, f = 6te^{-t}$.

$$W\left[e^{-t}, te^{-t}\right](t) = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & -te^{-t} + e^{-t} \end{vmatrix} = -te^{-2t} + e^{-2t} - (-te^{-2t}) = e^{-2t}$$

We let $y_p = v_1 y_1 + v_2 y_2 = v_1 e^{-t} + v_2 te^{-t}$ with

$$v_1 = \int \frac{-te^{-t}\left(6te^{-t}\right)}{e^{-2t}}\,\mathrm{d}t = -2t^3 \qquad v_2 = \int \frac{e^{-t}\left(6te^{-t}\right)}{e^{-2t}}\,\mathrm{d}t = 3t^2$$

so that $y_p = -2t^3 e^{-t} + 3t^2\left(te^{-t}\right) = t^3 e^{-t}$. The Nonhomogeneous Principle yields the general solution as

$$x(t) = c_1 e^{-t} + c_2 te^{-t} + t^3 e^{-t} = e^{-t}\left(c_1 + c_2 t + t^3\right)$$

Applying the initial conditions we have

$$x(0) = e^0\left(c_1\right) = 9$$
$$\dot{x}(t) = e^{-t}\left(c_2 + 3t^2\right) - e^{-t}\left(9 + c_2 t + t^3\right)$$
$$\dot{x}(0) = c_2 - 9 = -19 \implies c_2 = -10$$

so that the equation of motion of the oscillator is $x(t) = e^{-t}\left(9 - 10t + t^3\right)$.

(c) Yes. $0 = e^{-0}\left[9 - 10(1) + 1^3\right]$.

■

3. [2360/111225 (12 pts)] Let $\mathbb{S}$ be the solution space of $xy''' - y'' + xy' - y = 0,\ x > 0$. Does $\mathbb{S} = \text{span}\{x, \sin x, \cos x\}$? Fully justify your answer.

SOLUTION:

Begin by checking if all functions are solutions to the differential equation.

$$x(x)''' - (x)'' + x(x)' - x = x(0) - 0 + x - x = 0 \quad \checkmark$$
$$x(\sin x)''' - (\sin x)'' + x(\sin x)' - \sin x = x(-\cos x) - (-\sin x) + x\cos x - \sin x = 0 \quad \checkmark$$
$$x(\cos x)''' - (\cos x)'' + x(\cos x)' - \cos x = x(\sin x) - (-\cos x) + x(-\sin x) - \cos x = 0 \quad \checkmark$$

Next, check for linear independence of the functions via the Wronskian

$$W[x, \sin x, \cos x](x) = \begin{vmatrix} x & \sin x & \cos x \\ 1 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix} = x(-1)^{1+1}\begin{vmatrix} \cos x & -\sin x \\ -\sin x & -\cos x \end{vmatrix} + 1(-1)^{2+1}\begin{vmatrix} \sin x & \cos x \\ -\sin x & -\cos x \end{vmatrix}$$
$$= x(-\cos^2 x - \sin^2 x) - [-\sin x\cos x - (-\sin x\cos x)] = -x \neq 0 \implies \text{functions are linearly independent}$$

So, we have three linearly independent solutions of a third order linear homogeneous differential equation, the solution space of which is three dimensional, so the functions form a basis for the solution space, $\mathbb{S}$, with $\mathbb{S} = \text{span}\{x, \sin x, \cos x\}$. ■

4. [2360/111225 (28 pts)] Consider the initial value problem $y'' - 4y = 9(t-1)e^t,\ y(0) = y'(0) = 0$.

(a) (8 pts) Show that $\mathscr{L}\left\{(t-1)e^t\right\} = \dfrac{2-s}{(s-1)^2}$.

(b) (20 pts) Solve the given initial value problem using Laplace transforms. No points will be awarded for using any other technique.

SOLUTION:

(a) Alternative 1:

$$\mathscr{L}\left\{(t-1)e^t\right\} = \mathscr{L}\left\{te^t\right\} - \mathscr{L}\left\{e^t\right\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s-1}\right) - \frac{1}{s-1}$$

$$= \frac{1}{(s-1)^2} - \frac{1}{s-1} = \frac{1}{(s-1)^2} - \frac{s-1}{(s-1)^2} = \frac{2-s}{(s-1)^2}$$

Alternative 2:

$$\mathscr{L}\left\{(t-1)e^t\right\} = \mathscr{L}\left\{te^t\right\} - \mathscr{L}\left\{e^t\right\} = \mathscr{L}\left\{t\right\}\Big|_{s\to s-1} - \frac{1}{s-1}$$

$$= \frac{1}{s^2}\Big|_{s\to s-1} - \frac{1}{s-1} = \frac{1}{(s-1)^2} - \frac{1}{s-1} = \frac{1}{(s-1)^2} - \frac{s-1}{(s-1)^2} = \frac{2-s}{(s-1)^2}$$

(b)

$$\mathscr{L}\left\{y'' - 4y\right\} = s^2Y(s) - sy(0) - y'(0) - 4Y(s) = \mathscr{L}\left\{9(t-1)e^t\right\} = \frac{9(2-s)}{(s-1)^2}$$

$$Y(s) = \frac{9(2-s)}{(s-1)^2\,(s^2-4)}$$

Now

$$\frac{9(2-s)}{(s-1)^2\,(s^2-4)} = \frac{9(2-s)}{(s-1)^2(s+2)(s-2)} = \frac{-9}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$$

$$-9 = A(s-1)(s+2) + B(s+2) + C(s-1)^2$$

$$s = 1: -9 = B(3) \implies B = -3$$

$$s = -2: -9 = C(-3)^2 \implies C = -1$$

$$s = 0: -9 = A(-1)(2) - 3(2) - (-1)^2 \implies A = 1$$

Then

$$Y(s) = \frac{9(2-s)}{(s-1)^2\,(s^2-4)} = \frac{1}{s-1} - \frac{3}{(s-1)^2} - \frac{1}{s+2}$$

$$y(t) = \mathscr{L}^{-1}\left\{Y(s)\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s-1} - \frac{3}{(s-1)^2} - \frac{1}{s+2}\right\} = e^t - 3te^t - e^{-2t} = e^t(1-3t) - e^{-2t}$$

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5. [2360/111225 (24 pts)] Each part [(a)-(f)] below consists of a characteristic equation obtained from a linear, constant coefficient, homogeneous differential equation $\mathcal{L}(\vec{\mathbf{y}}) = 0$ along with a forcing function, $f(t)$, from the nonhomogeneous equation $\mathcal{L}(\vec{\mathbf{y}}) = f(t)$. Choose the one function from the boxed functions below (just write a number I-X) that would be used in the method of undetermined coefficients to solve $\mathcal{L}(\vec{\mathbf{y}}) = f(t)$. DO NOT find any of the constants.

To help with grading, please place your numbers in a single column away from any work you may perform to arrive at the answer. No work need be shown and no partial credit will be given. Not all of the functions in the box will be used.

I. $At^5 + Bt^4 + Ct^3$	II. $(At+B)\cos 3t + (Ct+D)\sin 3t$	III. $A\cos 3t + B\sin 2t$	IV. $At + Be^t$
V. $At + B\sin t + C\cos t$	VI. $Ae^t + B$	VII. $A\cos 2t + B\sin 2t$	VIII. $\left(At^2 + Bt\right)e^{-3t}$
IX. $te^t\left(A\cos t + B\sin t\right) + C$	X. $At\cos 3t + Bt\sin 3t$		

(a) $2r^2 + 5r - 3 = 0;\quad f(t) = te^{-3t}$

(b) $r^3 = 0;\quad f(t) = t^2$

(c) $r^2 - 4r + 8 = 0;\quad f(t) = \cos 2t$

(d) $r^2 + r = 0;\quad f(t) = 2 + e^t$

(e) $r^2 + 6r + 9 = 0; \quad f(t) = t \sin 3t$

(f) $r^3 - 2r^2 + 2r = 0; \quad f(t) = \cos t + \sin t - 7$

SOLUTION:

(a) **VIII** Basis for solution space of homogeneous equation is $\left\{e^{t/2}, e^{-3t}\right\}$; initial guess $y_p = (At + B)e^{-3t}$; modified guess $y_p = \left(At^2 + Bt\right) e^{-3t}$

(b) **I** Basis for solution space of homogeneous equation is $\left\{1, t, t^2\right\}$; initial guess $y_p = At^2 + Bt + C$; modified guess $y_p = At^5 + Bt^4 + Ct^3$

(c) **VII** Basis for solution space of homogeneous equation is $\left\{e^{2t} \cos 2t, e^{2t} \sin 2t\right\}$; initial guess $y_p = A \cos 2t + B \sin 2t$; no modification necessary

(d) **IV** Basis for solution space of homogeneous equation is $\{1, e^{-t}\}$; initial guess $y_p = A + Be^t$; modified guess $y_p = At + Be^t$

(e) **II** Basis for solution space of homogeneous equation is $\left\{e^{-3t}, te^{-3t}\right\}$; initial guess $y_p = (At + B) \cos 3t + (Ct + D) \sin 3t$; no modification necessary

(f) **V** Basis for solution space of homogeneous equation is $\{1, e^t \cos t, e^t \sin t\}$; initial guess $y_p = A \cos t + B \sin t + C$; modified guess $y_p = A \cos t + B \sin t + Ct$

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