1. (16 pts) Use Lagrange multipliers to find the point(s) on the hyperbola $x^2 - y^2 = 1$ that are closest to the point $(0, 4)$. (***Hint:*** You may minimize the square of the distance.)

   Solution:

   The distance from a point (x, y) to the point $(0, 4)$ is

   $$d(x, y) = \sqrt{x^2 + (y-4)^2}.$$

   We will minimize the square of the distance

   $$f(x, y) = x^2 + (y-4)^2$$

   given the constraint

   $$g(x, y) = x^2 - y^2 = 1.$$

   The gradient vectors are

   $$\nabla f = \langle 2x, 2y-8 \rangle \text{ and } \nabla g = \langle 2x, -2y \rangle.$$

   Set $\nabla f = \lambda \nabla g$:

   $$\begin{aligned} \langle 2x, 2y-8 \rangle &= \lambda \langle 2x, -2y \rangle \\ 2x &= \lambda(2x) \\ 2y - 8 &= \lambda(-2y). \end{aligned}$$

   The first equation implies $x = 0$ or $\lambda = 1$. Because $x = 0$ does not intersect the hyperbola, λ must equal 1. Then $2y - 8 = -2y \implies y = 2$.

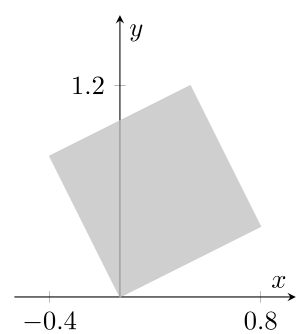
   Substituting $y = 2$ into the constraint gives

   $$x^2 - 2^2 = 1 \implies x = \pm\sqrt{5}.$$

   Therefore the two points on the hyperbola closest to $(0, 4)$ are $(\pm\sqrt{5}, 2)$.

2. (16 pts) Consider the integral $\iint_R (2x+y)^2 \sqrt{x-y}\, dx\, dy$, where R, shown at right, is bounded by

   $$y = \frac{x}{2},\ \ y = \frac{x}{2} + 1,\ \ y = -2x,\ \ y = -2x + 2.$$



   (a) Let $u = 2x + y$ and $v = x - y$. Sketch the transformed region in the uv-plane.

   (b) Set up (but do not evaluate) an equivalent uv-integral.

   Solution:

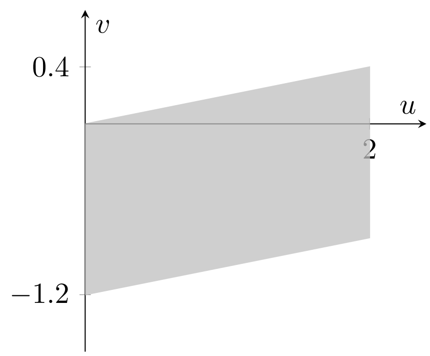
   (a) If $u = 2x + y$ and $v = x - y$, then adding the two equations gives $x = \dfrac{u+v}{3}$.

   Multiplying the second equation by 2, then subtracting the two equations, gives $y = \dfrac{u-2v}{3}$.

   The new boundaries are shown in the table below.

xy boundaries	uv boundaries	uv equations
$y = \frac{x}{2}$	$\frac{u-2v}{3} = \frac{u+v}{6}$	$u = 5v$
$y = \frac{x}{2} + 1$	$\frac{u-2v}{3} = \frac{u+v}{6} + 1$	$u = 5v + 6$
$y = -2x$	$\frac{u-2v}{3} = \frac{-2(u+v)}{3}$	$u = 0$
$y = -2x + 2$	$\frac{u-2v}{3} = \frac{-2(u+v)}{3} + 2$	$u = 2$

The corresponding uv region is shown below.



(b) The Jacobian is

$$J(u,v) = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{vmatrix} = -\frac{1}{3}.$$

Therefore an equivalent integral in the uv-plane is

$$\int_0^2 \int_{(u-6)/5}^{u/5} \frac{1}{3} u^2 \sqrt{v}\, dv\, du.$$

3. (18 pts) Evaluate $\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2}\, dx\, dy$ by converting to a polar double integral.

Solution:

The region of integration is the first quadrant of the xy-plane, so $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq r < \infty$. Substitute $r^2 = x^2 + y^2$. The Jacobian of the transformation is r.

$$\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2}\, dx\, dy = \int_0^{\pi/2} \int_0^\infty \frac{r}{(1+r^2)^2}\, dr\, d\theta$$

Let $u = 1 + r^2$, $du = 2r\, dr$.

$$= \left(\int_0^{\pi/2} d\theta \right) \left(\int_1^\infty \frac{1}{2u^2}\, du \right)$$

$$= \left([\theta]_0^{\frac{\pi}{2}} \right) \left(\lim_{t\to\infty} \left[-\frac{1}{2u} \right]_1^t \right)$$

$$= \frac{\pi}{2} \left(0 + \frac{1}{2} \right) = \frac{\pi}{4}$$

4. (28 pts) Consider the solid with volume $V = \int_0^{\pi} \int_0^5 \int_r^5 r\, dz\, dr\, d\theta$ in cylindrical coordinates.

   (a) Sketch and shade a cross-section of the solid in the rz–plane (that is, a half-plane of constant θ). Label the intercepts.

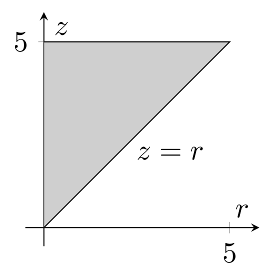
   (b) Set up (but do not evaluate) an equivalent integral using

      i. rectangular coordinates in the order $dz\, dy\, dx$

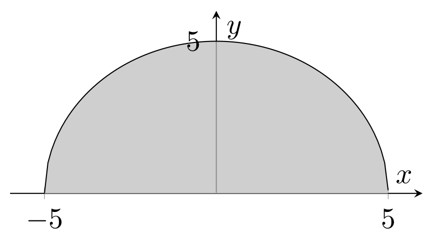
      ii. spherical coordinates in the order $d\rho\, d\phi\, d\theta$.

Solution:

(a) The solid is a filled cone above the xy-plane. Below is a cross-section of the solid in the rz-plane.



(b) i. The projection of the cone onto the xy-plane is a semicircular region of radius 5 above the x-axis.



In the z direction, the solid extends from the cone $z = r = \sqrt{x^2 + y^2}$ to the plane $z = 5$, so the integral in rectangular coordinates is

$$\int_{-5}^{5} \int_0^{\sqrt{25-x^2}} \int_{\sqrt{x^2+y^2}}^{5} dz\, dy\, dx.$$

ii. In spherical coordinates, the angle ϕ corresponding to $z = r$ is $\frac{\pi}{4}$. The plane $z = \rho \cos\phi = 5$ corresponds to $\rho = 5 \sec\phi$. In the semicircular region, the angle θ ranges from 0 to π. Therefore the integral in spherical coordinates is

$$\int_0^{\pi} \int_0^{\pi/4} \int_0^{5\sec\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

5. The following two problems are not related.

(a) (12 pts) Let X and Y be continuous random variables with joint probability density function

$$f(x,y) = \begin{cases} k & \text{if } 0 \le x \le 50,\ 10 \le y \le 25 \\ 0 & \text{otherwise.} \end{cases}$$

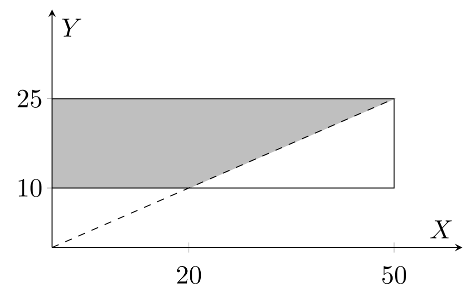
i. Find the constant k.

ii. Set up (but do not evaluate) a double integral that represents $P\left(Y \ge \frac{X}{2}\right)$.

Solution:

i. The rectangular region with width 50 and height 15 has area 750, so $k = 1/750$.

ii. The probability corresponds to the region to the left of the line $y = \frac{x}{2}$. An integral representation is $\displaystyle\int_{10}^{25}\int_{0}^{2y} \frac{1}{750}\,dx\,dy$.



(b) (10 pts) A matrix A is *skew-symmetric* if $A = -A^T$.

i. Give an example of a 3×3 matrix B with nonzero entries that is skew-symmetric.

ii. Compute $B - B^T$.

Solution:

i. Here is one example:

$$B = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -5 \\ -3 & 5 & 0 \end{bmatrix}.$$

ii.

$$B - B^T = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -5 \\ -3 & 5 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 & 6 \\ 4 & 0 & -10 \\ -6 & 10 & 0 \end{bmatrix}$$

The matrix $B - B^T = B + B = 2B$ which also is skew-symmetric.