- This exam is worth 100 points and has 5 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5”× 11” crib sheet with writing on one side.

0. At the top of the page containing your solution to problem 1, write the following statement and sign your name to it: “I will abide by the CU Boulder Honor Code on this exam.” **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/111225 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

   (a) The equation of motion for any damped harmonic oscillator, forced or not, will always have a transient part.

   (b) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\} = e^{-t}\cos t$

   (c) The equation of motion for an undamped, unforced oscillator having a mass of 16 kg, restoring (spring) constant of 4 N/m, initially displaced 1 m to the left of the equilibrium position and given a velocity of 0.5 m/s to the left can be written as either $x(t) = -\cos\dfrac{t}{2} - \sin\dfrac{t}{2}$ or $x(t) = \sqrt{2}\cos\left(\dfrac{t}{2} - \dfrac{\pi}{4}\right)$.

   (d) The Euler-Cauchy differential equation $at^2y'' + bty' + cy = 0,\ t > 0$ where a, b, c are real numbers, can be written in the form $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\vec{\mathbf{x}}$ is a column vector and $\mathbf{A}$ is a 2×2 matrix.

   (e) The potential energy of the conservative system $2\ddot{y} + ye^y = 0$ is $e^y(y-1)$.

2. [2360/111225 (26 pts)] An harmonic oscillator consists of a 2 kg mass attached to a spring with restoring (spring) constant of 2 N/m. The damping force is numerically equal to four times the instantaneous velocity. The oscillator is subject to an external force equal to $12te^{-t}$. Initially, the mass is located 9 m to the right of the equilibrium position and provided with a velocity of 19 m/s to the left.

   (a) (3 pts) Is the oscillator overdamped, underdamped, undamped or critically damped? Justify your answer.

   (b) (20 pts) Use variation of parameters to find the equation of motion of the oscillator. No points will be awarded for using any other technique.

   (c) (3 pts) Is the mass at the equilibrium position when $t = 1$? Justify your answer.

3. [2360/111225 (12 pts)] Let $\mathbb{S}$ be the solution space of $xy''' - y'' + xy' - y = 0,\ x > 0$. Does $\mathbb{S} = \text{span}\{x, \sin x, \cos x\}$? Fully justify your answer.

4. [2360/111225 (28 pts)] Consider the initial value problem $y'' - 4y = 9(t-1)e^t,\ y(0) = y'(0) = 0$.

   (a) (8 pts) Show that $\mathscr{L}\{(t-1)e^t\} = \dfrac{2-s}{(s-1)^2}$.

   (b) (20 pts) Solve the given initial value problem using Laplace transforms. No points will be awarded for using any other technique.

MORE PROBLEMS AND LAPLACE TRANSFORM TABLE BELOW/ON REVERSE

5. [2360/111225 (24 pts)] Each part [(a)-(f)] below consists of a characteristic equation obtained from a linear, constant coefficient, homogeneous differential equation $\mathcal{L}(\vec{\mathbf{y}}) = 0$ along with a forcing function, $f(t)$, from the nonhomogeneous equation $\mathcal{L}(\vec{\mathbf{y}}) = f(t)$. Choose the one function from the boxed functions below (just write a number I-X) that would be used in the method of undetermined coefficients to solve $\mathcal{L}(\vec{\mathbf{y}}) = f(t)$. DO NOT find any of the constants.

To help with grading, please place your numbers in a single column away from any work you may perform to arrive at the answer. No work need be shown and no partial credit will be given. Not all of the functions in the box will be used.

I. $At^5 + Bt^4 + Ct^3$	II. $(At + B)\cos 3t + (Ct + D)\sin 3t$	III. $A\cos 3t + B\sin 2t$	IV. $At + Be^t$
V. $At + B\sin t + C\cos t$	VI. $Ae^t + B$	VII. $A\cos 2t + B\sin 2t$	VIII. $\left(At^2 + Bt\right)e^{-3t}$
IX. $te^t\,(A\cos t + B\sin t) + C$	X. $At\cos 3t + Bt\sin 3t$		

(a) $2r^2 + 5r - 3 = 0; \quad f(t) = te^{-3t}$

(b) $r^3 = 0; \quad f(t) = t^2$

(c) $r^2 - 4r + 8 = 0; \quad f(t) = \cos 2t$

(d) $r^2 + r = 0; \quad f(t) = 2 + e^t$

(e) $r^2 + 6r + 9 = 0; \quad f(t) = t\sin 3t$

(f) $r^3 - 2r^2 + 2r = 0; \quad f(t) = \cos t + \sin t - 7$

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2 + b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2 + b^2}$$

$$\mathscr{L}\{\cosh bt\} = \frac{s}{s^2 - b^2} \qquad \mathscr{L}\{\sinh bt\} = \frac{b}{s^2 - b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{tf'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\text{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\text{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$