1. [2350/111225 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

   (a) The vector field $\mathbf{V} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$ is incompressible but not irrotational.

   (b) The following all represent the same point in space:

   $$(x, y, z) = \left(-\frac{3\sqrt{2}}{4}, -\frac{3\sqrt{2}}{4}, \frac{\sqrt{3}}{2}\right) \qquad (\rho, \theta, \phi) = \left(\sqrt{3}, \frac{\pi}{4}, \frac{\pi}{3}\right) \qquad (r, \theta, z) = \left(\frac{3}{2}, \frac{\pi}{4}, \frac{\sqrt{3}}{2}\right)$$

   (c) The area of a certain region in the xy-plane is given equivalently by $\displaystyle\int_0^1 \int_1^{1+\sqrt{1-x^2}} \mathrm{d}y\,\mathrm{d}x$ or $\displaystyle\int_{\pi/4}^{\pi/2} \int_{\csc\theta}^{2\sin\theta} r\,\mathrm{d}r\,\mathrm{d}\theta$

   (d) If $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $r = \|\mathbf{r}\|$, then $\nabla \cdot (r\mathbf{r}) = 4r$.

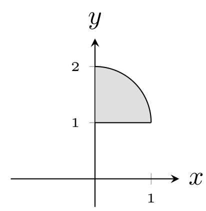
   (e) If $f(x, y, z)$ is a scalar field possessing continuous second derivatives and $\mathbf{F}(x, y, z)$ is a vector field, both $\nabla \times (\nabla f)$ and $\nabla(\nabla \times \mathbf{F})$ are valid operations.

SOLUTION:

(a) **FALSE** $\nabla \cdot \mathbf{V} = 0$, making the field incompressible. However, since $\nabla \times \mathbf{V} = \mathbf{0}$, the vector field is also irrotational.

(b) **FALSE** The spherical and cylindrical coordinate points are above the incorrect quadrant. θ should be $5\pi/4$ in both cases.

(c) **TRUE** Here is the region under consideration.



(d) **TRUE**

$$\nabla \cdot (r\mathbf{r}) = r\nabla \cdot \mathbf{r} + \mathbf{r} \cdot \nabla r = \sqrt{x^2 + y^2 + z^2}(3) + \langle x, y, z\rangle \cdot \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{x}{\sqrt{x^2+y^2+z^2}} \right\rangle$$

$$= 3r + \frac{x^2 + y^2 + z^2}{\sqrt{x^2 + y^2 + z^2}} = 3r + \sqrt{x^2 + y^2 + z^2} = 3r + r = 4r$$

(e) **FALSE** The first operation evaluates to $\mathbf{0}$ (gradient vector fields are irrotational). The second operation is not possible (only scalar fields possess gradients).

■

2. [2350/111225 (28 pts)] Buzz Lightyear has a cool new shield to protect himself from Emperor Zurg and he needs to know the shield's area. It can be modeled as the region above the line $y = 2x$, $0 \le x \le 1$ and below the function $f(x, y) = 1 + x$.

   (a) (14 pts) Use a scalar line integral to compute the area of the shield for Buzz. Tom Sawyer and his aunt did a similar calculation in the homework.

   (b) (14 pts) Star Command is requesting that Buzz verify his calculation from part (a) by using a scalar surface integral. They have told him that the shield can be parameterized as $\mathbf{r}(u, v) = \langle u, 2u, v(1 + u)\rangle$ with the parameter range $0 \le u \le 1,\ 0 \le v \le 1$. Please fulfill Star Command's request by showing Buzz that your answer here agrees with that from part (a).

SOLUTION:

(a) The area is given by $\int_{\mathcal{C}} (1+x)\,\mathrm{d}s$. The curve, $\mathcal{C}$, is a function, parameterized simply as $\mathbf{r}(t) = \langle t, 2t \rangle, 0 \le t \le 1$. Then

$$\mathbf{r}'(t) = \langle 1, 2 \rangle \implies \|\mathbf{r}'(t)\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$f(\mathbf{r}(t)) = 1 + t$$

$$\text{Area} = \int_{\mathcal{C}} (1+x)\,\mathrm{d}s = \int_0^1 f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\,\mathrm{d}t = \int_0^1 (1+t)\sqrt{5}\,\mathrm{d}t = \sqrt{5}\,\frac{(1+t)^2}{2}\Big|_0^1 = \frac{3\sqrt{5}}{2}$$

(b) The area now is given by $\iint_{\mathcal{S}} \mathrm{d}S$ where $\mathcal{S}$ represents the shield. We have

$$\mathbf{r}(u,v) = \langle u, 2u, v(1+u) \rangle \qquad \mathcal{R} = \left\{ (u,v) \in \mathbb{R}^2 \,\middle|\, 0 \le u \le 1, 0 \le v \le 1 \right\}$$

$$\mathbf{r}_u = \langle 1, 2, v \rangle, \quad \mathbf{r}_v = \langle 0, 0, 1+u \rangle \implies \mathbf{r}_u \times \mathbf{r}_v = \langle 2(1+u), -(1+u), 0 \rangle$$

$$\implies \|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{4(1+u)^2 + [-(1+u)]^2} = \sqrt{5}\,|1+u| = \sqrt{5}\,(1+u)$$

$$\text{Area} = \iint_{\mathcal{S}} \mathrm{d}S = \iint_{\mathcal{R}} \|\mathbf{r}_u \times \mathbf{r}_v\|\,\mathrm{d}A = \int_0^1 \int_0^1 \sqrt{5}\,(1+u)\,\mathrm{d}u\,\mathrm{d}v = \sqrt{5} \int_0^1 \frac{(1+u)^2}{2}\Big|_0^1 \mathrm{d}v = \sqrt{5} \int_0^1 \frac{3}{2}\,\mathrm{d}v = \frac{3\sqrt{5}}{2}$$

Star Command's request is fulfilled.

3. [2350/111225 (24 pts)] Unlike the previous exams, this time there is a whole bunch of hummingbirds. The density of the little creatures is $h(x,y,z) = z\sqrt{x^2+y^2}$ hummingbirds per cubic foot. A new bird sanctuary has just opened and the special area set aside for the hummingbirds is the region above the second quadrant bounded by the cones $z = \sqrt{x^2+y^2}$ and $z = \sqrt{3\,(x^2+y^2)}$ and between the cylinders $x^2+y^2 = 1$ and $x^2+y^2 = 4$. In each part below, set up, but **DO NOT EVALUATE**, integral(s) that will compute the total number of hummingbirds that visitors will experience when viewing the special area. Use the given coordinates and order of integration provided in each part. Be sure to simplify your integrands and use bounds that describe the region exactly as given without exploiting any symmetries.

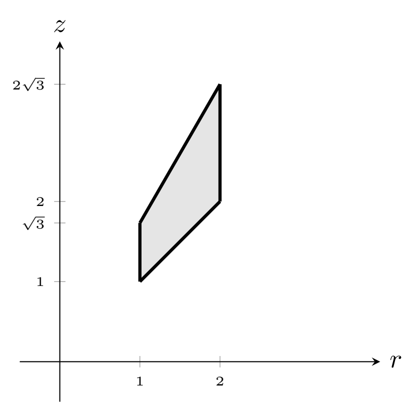
(a) (8 pts) Cylindrical coordinates; $\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$

(b) (8 pts) Spherical coordinates; $\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$

(c) (8 pts) Rectangular/Cartesian coordinates; $\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$

SOLUTION:

Here is a sketch of the region in a constant θ plane.



(a)

$$\int_{\pi/2}^{\pi} \int_1^2 \int_r^{\sqrt{3}r} zr^2\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

(b)

$$\int_{\pi/2}^{\pi} \int_{\pi/6}^{\pi/4} \int_{\csc\phi}^{2\csc\phi} \rho^4 \sin^2\phi \cos\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

(c)

$$\int_{-2}^{-1}\int_{0}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{3(x^2+y^2)}} z\sqrt{x^2+y^2}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x + \int_{-1}^{0}\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{3(x^2+y^2)}} z\sqrt{x^2+y^2}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

■

4. [2350/111225 (10 pts)] Evaluate $\int_{\mathcal{C}} (x+y+z)\,\mathrm{d}x + x\,\mathrm{d}y - yz\,\mathrm{d}z$ where $\mathcal{C}$ is the line segment from $(1,2,1)$ to $(2,1,0)$.

SOLUTION:

$$\mathcal{C} : \mathbf{r}(t) = (1-t)\langle 1,2,1\rangle + t\langle 2,1,0\rangle = \langle 1+t, 2-t, 1-t\rangle,\ 0 \le t \le 1$$

$$x = 1+t \implies \mathrm{d}x = \mathrm{d}t$$

$$y = 2-t \implies \mathrm{d}y = -\mathrm{d}t$$

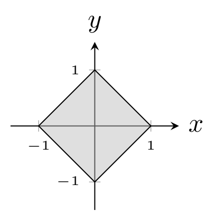
$$z = 1-t \implies \mathrm{d}z = -\mathrm{d}t$$

$$\int_{\mathcal{C}} (x+y+z)\,\mathrm{d}x + x\,\mathrm{d}y - yz\,\mathrm{d}z = \int_0^1 (1+t+2-t+1-t)\,\mathrm{d}t + (1+t)(-\mathrm{d}t) - (2-t)(1-t)(-\mathrm{d}t)$$

$$= \int_0^1 \left(5 - 5t + t^2\right)\mathrm{d}t = \left(5t - \frac{5}{2}t + \frac{t^3}{3}\right)\Big|_0^1 = 5 - \frac{5}{2} + \frac{1}{3} = \frac{17}{6}$$

■

5. [2350/111225 (18 pts)] A thin plate in the shape $(\mathcal{R})$ shown below is made of material having a density of $\rho(x,y) = e^{x+y}$. We need to find the mass of the plate.



(a) (4 pts) Without exploiting any symmetries or formally setting up any integrals, state the number of integrals required to do this mass calculation directly using $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$ and again using $\mathrm{d}A = \mathrm{d}y\,\mathrm{d}x$.

(b) (14 pts) Now compute the mass of the plate using the change of variables $u = x+y, v = x-y$.

SOLUTION:

(a) Both orders of integration require 2 integrals, the top and bottom halves for $\mathrm{d}x\,\mathrm{d}y$ and the left and right halves for $\mathrm{d}y\,\mathrm{d}x$.

(b) Using the given transformation we have

$$x = \frac{1}{2}(u+v),\ y = \frac{1}{2}(u-v) \quad J(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

The boundaries of the plate consisting of the lines with positive slope are $y = x+1$ or $x - y = -1$ and $y = x-1$ or $x - y = 1$. This implies $-1 \le v \le 1$. The other boundaries are $y = -x-1$ or $x+y = 1$ and $y = -x-1$ or $x+y = -1$. These imply $-1 \le u \le 1$ and we have the new region of integration, $\mathcal{D}$. The density is $\rho(u,v) = e^u$. Thus,

$$\text{Mass} = \iint_{\mathcal{R}} \rho(x,y)\,\mathrm{d}A = \iint_{\mathcal{D}} \rho(u,v)|J(u,v)|\,\mathrm{d}u\,\mathrm{d}v = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} e^u\,\mathrm{d}u\,\mathrm{d}v$$

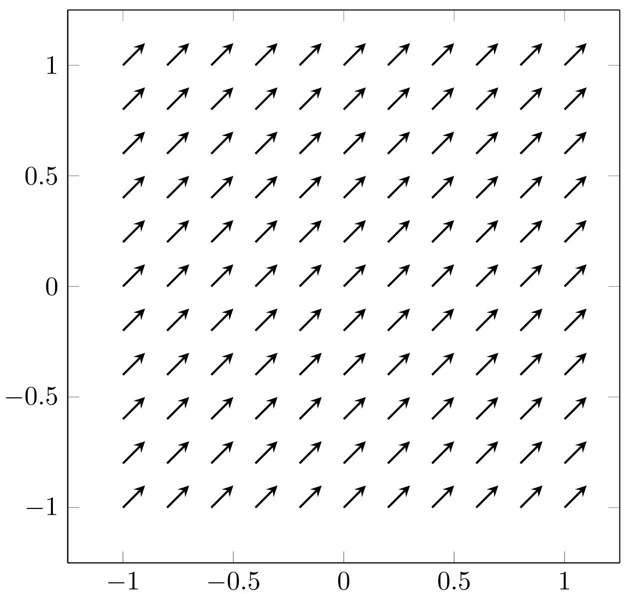
$$= \frac{1}{2}\int_{-1}^{1} e^u\Big|_{-1}^{1}\mathrm{d}v = \frac{1}{2}\left(e - e^{-1}\right)\int_{-1}^{1}\mathrm{d}v = e - e^{-1}$$

■

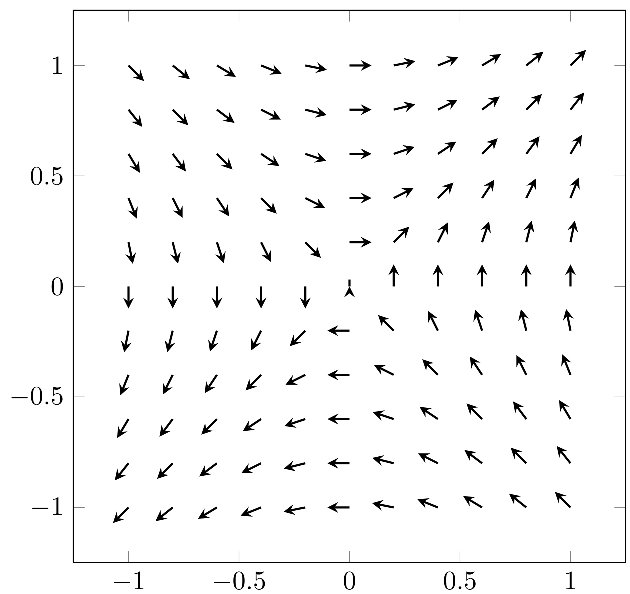
6. [2350/111225 (10 pts)] Match the following vector fields to the appropriate graph. Write NONE as needed.

(a) $\mathbf{i} + x\,\mathbf{j}$ (b) $\langle x+y, x-y\rangle$ (c) $\langle 1, 1\rangle$ (d) $x\,\mathbf{i} + y\,\mathbf{j}$ (e) $\dfrac{1}{\sqrt{x^2+y^2}}\langle y, x\rangle$

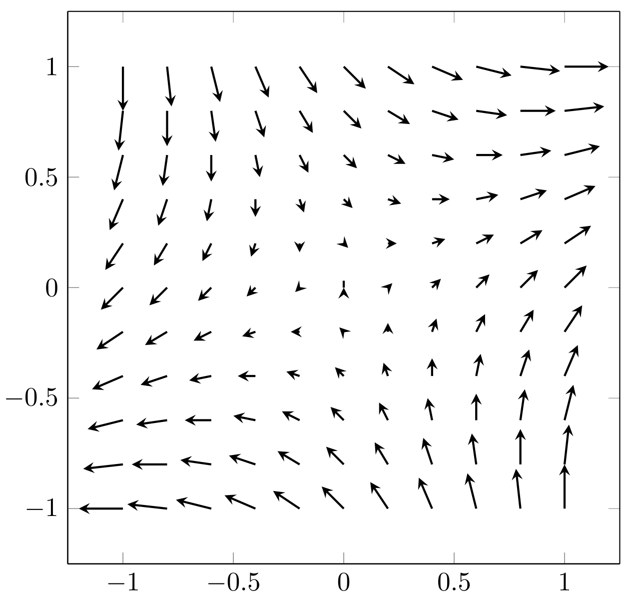
I



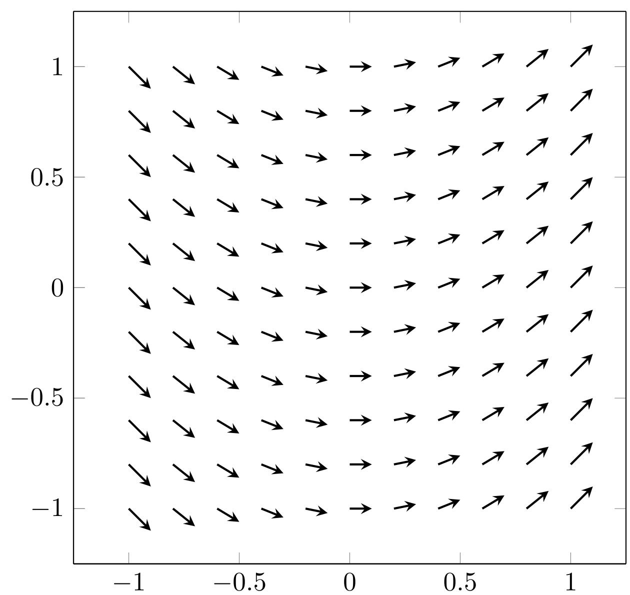
II



III



IV



SOLUTION:

(a) IV (b) III (c) I (d) NONE (e) II