1. (32 points) The following problems are not related.

   (a) Suppose $g'(x) = \sin x$ and $g(\pi/4) = 3$. Determine $g(x)$.

   (b) Evaluate $\displaystyle\int_1^4 \frac{5x^2 - 3x + \sqrt{x}}{\sqrt{x}}\,dx$.

   (c) Suppose the average value of $f(x)$ along $[-1, 5]$ is 3, and that $\displaystyle\int_2^5 f(x)\,dx = 8$. Find $\displaystyle\int_{-1}^2 f(x)\,dx$.

   (d) Evaluate $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(2 + \frac{3i}{n}\right)\cdot\frac{3}{n}\right]$ using any method discussed in this class.

Solution:

(a) The most general antiderivative of $g'(x) = \sin x$ is

$$g(x) = -\cos(x) + C$$

where C is an arbitrary constant. We see that

$$3 = g(\pi/4) = -\cos(\pi/4) + C = -\frac{\sqrt{2}}{2} + C,$$

which means $C = \frac{6+\sqrt{2}}{2}$. Thus,

$$g(x) = -\cos(x) + \frac{6+\sqrt{2}}{2}.$$

(b)

$$\begin{aligned}\int_1^4 \frac{5x^2 - 3x + \sqrt{x}}{\sqrt{x}}\,dx &= \int_1^4 5x^{3/2} - 3x^{1/2} + 1\,dx\\ &= \Big[2x^{5/2} - 2x^{3/2} + x\Big]_1^4\\ &= [2(32) - 2(8) + 4] - [2 - 2 + 1]\\ &= 51.\end{aligned}$$

(c) We have $\displaystyle\frac{1}{5-(-1)}\int_{-1}^5 f(x)\,dx = 3$, which tells us that $\displaystyle\int_{-1}^5 f(x)\,dx = 18$. Thus,

$$\int_{-1}^2 f(x)\,dx = \int_{-1}^5 f(x)\,dx - \int_2^5 f(x)\,dx = 18 - 8 = 10.$$

(d) We have

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(2 + \frac{3i}{n}\right)\cdot\frac{3}{n}\right] &= \lim_{n\to\infty}\left[\frac{6}{n}\sum_{i=1}^{n}1 + \frac{9}{n^2}\sum_{i=1}^{n}i\right]\\ &= \lim_{n\to\infty}\left[6 + \frac{9}{n^2}\cdot\frac{n(n+1)}{2}\right]\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\left[6+\frac{9}{2}+\frac{9}{2n}\right]\\
&= 6+\frac{9}{2}\\
&= \frac{21}{2}
\end{aligned}
$$

or

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(2+\frac{3i}{n}\right)\cdot\frac{3}{n}\right] &= \int_0^3 2+x\,dx\\
&= \left[2x+\frac{x^2}{2}\right]_0^3\\
&= 6+\frac{9}{2}\\
&= \frac{21}{2}.
\end{aligned}
$$

or

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(2+\frac{3i}{n}\right)\cdot\frac{3}{n}\right] &= \int_2^5 x\,dx\\
&= \left[\frac{x^2}{2}\right]_2^5\\
&= \frac{25}{2}-\frac{4}{2}\\
&= \frac{21}{2}.
\end{aligned}
$$

2. (22 points) Consider $f(x) = x^2 - 3x - 3$, which is referenced in each part of this problem.

   (a) i. Show that f satisfies the hypotheses of Rolle's Theorem on the interval $[-1, k]$ for some value of $k > -1$. What must k equal?

   ii. Determine the value(s) c that satisfy the conclusion of the Rolle's Theorem for $f(x)$ over $[-1, k]$, where k is the value you found in (i).

   (b) Suppose someone were to apply Newton's method in an attempt to solve $f(x) = 0$, but after choosing an initial guess of x_1, they find that x_2 does not exist. Based on this information, what are the possible values for their initial guess of x_1? (Remember, as with all problems on this exam, you need to justify your result.)

   (c) Approximate $\displaystyle\int_0^2 (f(x)+3)\,dx$ using a right Riemann sum (use righthand endpoints) with three rectangles of equal width. Please DO NOT simplify your final answer, but it should be in a form that could be directly input into a calculator.

Solution:

(a) i. Our function f is continuous on $[-1, k]$ and differentiable on $(-1, k)$ for all $k > -1$ because it is a polynomial. We also need $f(-1) = f(k)$, which we can use to determine k. We see that $f(-1) = 1$, so our equation becomes

$$\begin{aligned}
f(k) &= 1 \\
k^2 - 3k - 3 &= 1 \\
k^2 - 3k - 4 &= 0 \\
(k-4)(k+1) &= 0 \\
k &= -1, 4.
\end{aligned}$$

We see that $k = 4$ is the value we are looking for. That is, Rolle's Theorem applies to $f(x)$ on $[-1, 4]$.

ii. We need to find c in $(-1, 4)$ such that $f'(c) = 0$.

$$\begin{aligned}
f'(c) &= 2c - 3 = 0 \\
c &= \frac{3}{2}.
\end{aligned}$$

(b) We see that $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ will not exist if $f'(x_1) = 0$. So, they must have guessed $x_1 = \frac{3}{2}$.

(c) We have $\Delta x = \frac{2-0}{3} = \frac{2}{3}$, so

$$\begin{aligned}
\int_0^2 (f(x) + 3)\, dx &\approx R_3 \\
&= \left[f\left(\frac{2}{3}\right) + 3 + f\left(\frac{4}{3}\right) + 3 + f(2) + 3 \right] \cdot \frac{2}{3} \\
&= \left[\left(\frac{2}{3}\right)^2 - 3\left(\frac{2}{3}\right) + \left(\frac{4}{3}\right)^2 - 3\left(\frac{4}{3}\right) + 2^2 - 3\,(2) \right] \cdot \frac{2}{3}.
\end{aligned}$$

3. (12 points) We want to build a cylindrical tank with a capacity of 24π cubic meters. The cost of the material to make the top and bottom is \$3 per square meter, and the cost of the material to make the side is \$2 per square meter. What dimensions will minimize the cost of the tank? (Be sure to justify that the dimensions you have found minimize the cost.)

You may find the following two facts helpful:

- The volume of a cylinder is $V = \pi r^2 h$.
- The total surface area of a cylinder is $A = 2\pi r h + 2\pi r^2$.

Solution:

We have the volume as $V = \pi r^2 h = 24\pi$, and we want to minimize the cost:

$$\begin{aligned}
\text{cost } &= (\text{ cost of side }) + (\text{ cost of top/bottom }) \\
&= (2\pi r h \text{ square meters })(\text{ \$ 2 per square meter }) + (2\pi r^2 \text{ square meters })(\text{ \$ 3 per square meter }) \\
&= 4\pi r h + 6\pi r^2 \text{ dollars }.
\end{aligned}$$

From our previous observation about the volume, we see that $h = \frac{24}{r^2}$, which means the cost can be written as a function of just r:

$$C(r) = \frac{96\pi}{r} + 6\pi r^2.$$

We note that the reasonable domain for our function is $(0, \infty)$. So, we will next find any critical numbers on this domain.

$$C'(r) = 12\pi r - \frac{96\pi}{r^2} = 12\pi\left(r - \frac{8}{r^2}\right) = 12\pi \cdot \frac{r^3 - 8}{r}.$$

We see that $C'(r)$ exists on our domain and that $C'(r) = 0$ only if $r = 2$, so $r = 2$ is the only critical number.

Since $C'(r) < 0$ for $0 < r < 2$ and $C'(r) > 0$ for $r > 2$, then the First Derivative Test for Absolute Extrema tells us that $C(r)$ has an absolute minimum at $r = 2$.

The corresponding value of h is $h = \frac{24}{2^2} = 6$. So, the dimensions that minimize the cost are a radius of 2 meters and a height of 6 meters.

4. (18 points) Consider the function $g(x) = \dfrac{x^{1/3}}{x+2}$, which is referenced in each part of this problem. You may assume without proof that its derivative is given by $g'(x) = \dfrac{2(1-x)}{3x^{2/3}(x+2)^2}$.

   (a) Find all critical numbers of $g(x)$.

   (b) Determine all the local extrema of $g(x)$ and identify each as a local maximum or minimum. Your argument should justify that all of the local extrema have been found. Clearly indicate the x-coordinates where the local extrema occur.

   (c) Determine $\dfrac{d}{dx}\left(\sec(4x) + \displaystyle\int_1^{\cos x} g(t)\,dt\right)$.

   Solution:

   (a) Note that $g'(x) = 0$ when $x = 1$, so $x = 1$ is a critical number. We see that $g'(x)$ does not exist for $x = 0, -2$, but $x = -2$ is not in the domain of g, so $x = -2$ is not a critical number, but $x = 0$ is. That is, the critical numbers of g are $x = 0, 1$.

   (b) Local extrema can only occur at critical numbers. That is, $x = 0, 1$ are the only possible locations of local extrema. The only sign change in $g'(x)$ occurs at $x = 1$, where g' goes from positive to negative as x increases. By the first derivative test, there is a local maximum of $g(1) = 1/3$ at $x = 1$, and it is the only local extreme value.

   (c)
$$\begin{aligned}\frac{d}{dx}\left(\sec(4x) + \int_1^{\cos x} g(t)\,dt\right) &= 4\sec(4x)\tan(4x) + g(\cos x)(-\sin x)\\ &= 4\sec(4x)\tan(4x) + \frac{\cos^{1/3} x}{\cos x + 2}(-\sin x)\end{aligned}$$

5. (16 pts) Using the grid below, sketch the graph of a **single function**, $y = f(x)$ with each of the following characteristics. (Sketch dashed lines to indicate any asymptotes that are present. The concavity of your graph should be clear.)

f is continuous on its domain:$(-\infty, -2) \cup (-2, \infty)$,	$f(-5/2) = 0$
$f(-3/2) = 0$,	$f'(x) < 0$ only if x is in $(-\infty, -2) \cup (-2, -3/2)$
$f''(x) > 0$ only if $x > -2$,	$\lim\limits_{x\to-\infty} f(x) = 4$
$\lim\limits_{x\to\infty} [f(x) - (4x+4)] = 0$,	$\lim\limits_{x\to-2^-} f(x) = -\infty$

Solution: