1. (18 pts) Determine $\dfrac{dy}{dx}$ for each of the following. After fully differentiating, do *not* algebraically simplify your answers.

   (a) $y = f(x) = \dfrac{1+5x^2}{3x-1}$

   Solution:

$$\frac{d}{dx}\left[\frac{1+5x^2}{3x-1}\right] = \frac{(3x-1)\cdot \frac{d}{dx}\left[1+5x^2\right] - \left(1+5x^2\right)\cdot \frac{d}{dx}\left[3x-1\right]}{(3x-1)^2}$$

$$= \boxed{\frac{(3x-1)\cdot(10x) - \left(1+5x^2\right)\cdot(3)}{(3x-1)^2}}$$

   (b) $y = g(x) = \sin^3(x^4+1)$

   Solution:

$$\frac{d}{dx}\left[\sin^3(x^4+1)\right] = 3\sin^2(x^4+1)\cdot\frac{d}{dx}\left[\sin(x^4+1)\right]$$

$$= 3\sin^2(x^4+1)\cdot\cos(x^4+1)\cdot\frac{d}{dx}\left[x^4+1\right]$$

$$= \boxed{3\sin^2(x^4+1)\cdot\cos(x^4+1)\cdot(4x^3)}$$

2. (26 pts) Parts (a) and (b) are unrelated.

(a) The position function of Particle P is given by $s(t) = t - \sqrt{2}\sin t$, where $0 \leq t \leq \pi$, s is in meters, and t is in seconds.

i. Find Particle P's velocity function $v(t)$. Include the correct unit of measurement.

ii. Find the distance traveled by Particle P between $t = 0$ and $t = \pi$ seconds. Fully simplify your answer and include the correct unit of measurement.

Solution:

i.

$$s(t) = t - \sqrt{2}\sin t$$
$$v(t) = s'(t) = \boxed{1 - \sqrt{2}\cos t \text{ m/s}}$$

ii.

$$s'(t) = 0 = 1 - \sqrt{2}\cos t$$
$$\cos t = 1/\sqrt{2}, \quad 0 \leq t \leq \pi$$
$$t = \pi/4$$

Particle P changes direction at time $t = \pi/4$, so that the total distance traveled between $t = 0$ and $t = \pi$ is

$$D = |s(\pi/4) - s(0)| + |s(\pi) - s(\pi/4)|$$

$$s(0) = 0 - \sqrt{2}\sin(0) = 0$$
$$s(\pi/4) = \pi/4 - \sqrt{2}\sin(\pi/4) = \pi/4 - \sqrt{2} \cdot (1/\sqrt{2}) = \pi/4 - 1$$
$$s(\pi) = \pi - \sqrt{2}\sin(\pi) = \pi$$

$$D = |(\pi/4 - 1) - 0| + |\pi - (\pi/4 - 1)|$$
$$= (1 - \pi/4) + (3\pi/4 + 1) = \boxed{2 + \pi/2 \text{ m}}$$

(b) The velocity function of Particle Q is given by $v(t) = 2t^{1/2} - t^{3/2}$, where $t > 0$, v is in miles per hour, and t is in hours.

i. Find Particle Q's acceleration function $a(t)$. Include the correct unit of measurement.

ii. Is the velocity of Particle Q increasing, decreasing, or neither at $t = 1$? Briefly explain your answer.

Solution:

i.

$$\begin{aligned} v(t) &= 2t^{1/2} - t^{3/2} \\ a(t) &= v'(t) = 2 \cdot (1/2)t^{-1/2} - (3/2)t^{1/2} \\ &= \boxed{t^{-1/2} - (3/2)t^{1/2} \text{ miles / hr}^2} \end{aligned}$$

ii.

$$a(1) = 1^{-1/2} - (3/2) \cdot 1^{1/2} = 1 - 3/2 = -1/2 < 0$$

Since $a(t)$ for Particle Q is negative at $t = 1$, the velocity of particle Q is $\boxed{\text{decreasing}}$ at $t = 1$.

3. (22 pts) Parts (a) and (b) are unrelated.

(a) Identify all values of x, if any, at which $y = u(x) = x^{-1} - 2x^{-3}$ has a horizontal tangent. If no horizontal tangents exist, clearly state "none".

Solution:

$$\begin{aligned} u'(x) &= -x^{-2} - 2 \cdot (-3) \cdot x^{-4} \\ &= -x^{-2} + 6x^{-4} \\ &= x^{-4}(-x^2 + 6) \end{aligned}$$

The curve $y = u(x)$ has a horizontal tangent at all values of x such that $u'(x) = x^{-4}(-x^2 + 6) = 0$.

There is no value of x such that $x^{-4} = \dfrac{1}{x^4} = 0$.

$-x^2 + 6 = 0$ for $x = \boxed{\pm\sqrt{6}}$

(b) Find the equations of the tangent and normal lines to the curve $y = w(x) = \tan x + 3x^2 - 7x - 1$ at $x = 0$. If there are no tangent and normal lines at that location, clearly state "none".

Solution:

$$\begin{aligned} w'(x) &= \sec^2 x + 6x - 7 \\ w'(0) &= \sec^2(0) + 6 \cdot 0 - 7 \\ &= 1 - 7 = -6 \end{aligned}$$

$$w(0) = \tan(0) + 3 \cdot 0^2 - 7 \cdot 0 - 1 = -1$$

The point-slope form of the equation of the tangent line is $y - (-1) = -6(x - 0)$. Therefore,

Tangent line: $\boxed{y + 1 = -6x}$

Normal line: $\boxed{y + 1 = \dfrac{1}{6}x}$

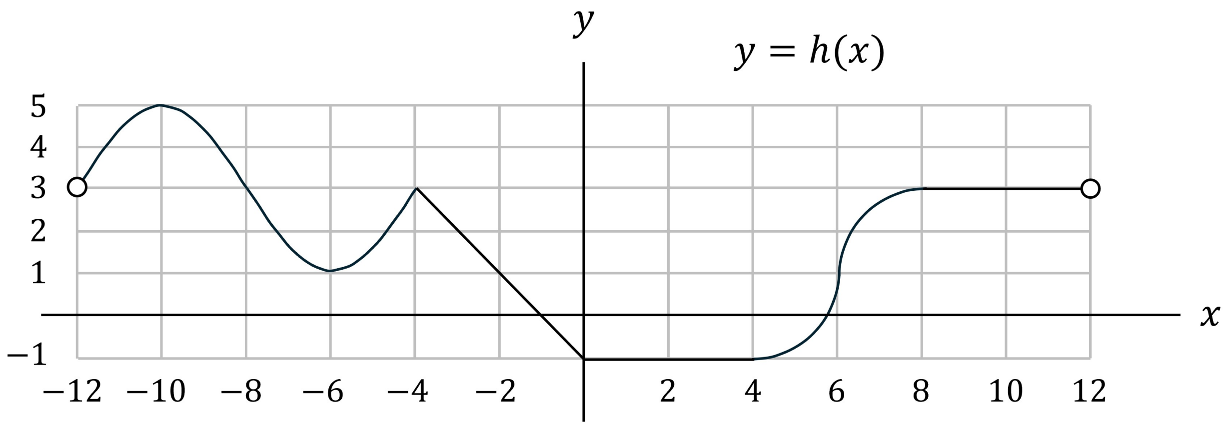
4. (21 pts) Parts (a) and (b) are unrelated.

(a) Determine $p'(x)$ for the function $p(x) = x^2 - x + 1$ by using the *definition of derivative*. You must obtain $p'(x)$ by evaluating the appropriate *limit*.

Solution:

$$\begin{aligned}
p'(x) &= \lim_{h\to 0} \frac{p(x+h) - p(x)}{h} \\
&= \lim_{h\to 0} \frac{((x+h)^2 - (x+h) + 1) - (x^2 - x + 1)}{h} \\
&= \lim_{h\to 0} \frac{(x^2 + 2hx + h^2 - x - h + 1) - (x^2 - x + 1)}{h} \\
&= \lim_{h\to 0} \frac{2hx + h^2 - h}{h} \\
&= \lim_{h\to 0} (2x + h - 1) = \boxed{2x - 1}
\end{aligned}$$

(b) The following is the graph of a function $h(x)$ on the domain $(-12, 12)$.



Answer the following, where $h'(x)$ is the derivative function of the function $h(x)$ graphed above.

i. What is the value of $h'(-10)$?

ii. What is the value of $h'(-2)$?

iii. On what subintervals of $(-12, 12)$, if any, is the value of $h'(x)$ negative? If no such subintervals exist, clearly state "none".

iv. At what values of x on the interval $(-12, 12)$, if any, is $h(x)$ not differentiable? If no such x values exist, clearly state "none".

Solution:

i. Because $y = h(x)$ has a horizontal tangent at $x = -10$, we know that $h'(-10) = \boxed{0}$

ii. The graph of $y = h(x)$ is linear on the interval $[-4, -2]$. Its slope can be calculated using the coordinates of its endpoints, $(-4, 3)$ and $(0, -1)$.

$$\frac{\Delta y}{\Delta x} = \frac{-1-3}{0-(-4)} = \frac{-4}{4} = -1$$

Since the slope of $y = h(x)$ equals -1 at $x = -2$, we know that $h'(-2) = \boxed{-1}$

iii. The value of $h'(x)$ is negative at any x value at which the slope of the tangent line to $y = f(x)$ is negative. The graph indicates that the subtintervals of x on which that occurs are $\boxed{(-10, -6) \text{ and } (-4, 0)}$

iv. $h(x)$ is not differentiable at $\boxed{x = -4}$ ($y = h(x)$ has a corner there), $\boxed{x = 0}$ ($y = h(x)$ has a corner there), and $\boxed{x = 6}$ ($y = h(x)$ has a vertical tangent there).

5. (13 pts) Find the equation of each horizontal asymptote of $y = \dfrac{(x+2)(3x-5)}{(4-x)(2x+3)}$, if any exist. If no horizontal asymptotes exist, clearly state "none".

Support your answer by evaluating the appropriate limit(s). *(Reminder: You may not use L'Hôpital's Rule or dominance of powers arguments to evaluate limits on this exam.)*

Solution:

$$\begin{aligned} y &= \frac{(x+2)(3x-5)}{(4-x)(2x+3)} \\ &= \frac{3x^2+x-10}{-2x^2+5x+12} \end{aligned}$$

$$\begin{aligned} \lim_{x\to\pm\infty} \frac{3x^2+x-10}{-2x^2+5x+12} &= \lim_{x\to\pm\infty} \frac{3x^2+x-10}{-2x^2+5x+12} \cdot \frac{1/x^2}{1/x^2} \\ &= \lim_{x\to\pm\infty} \frac{3+1/x-10/x^2}{-2+5/x+12/x^2} \\ &= \frac{3+0-0}{-2+0+0} = -\frac{3}{2} \end{aligned}$$

Therefore, the equation of the only horizontal asymptote is $\boxed{y = -3/2}$