Potentially useful formula: Let u and w denote positive real numbers, then:

$$\log_b(u) = \frac{\log_a(u)}{\log_a(b)} \text{ for } a > 0,\ a \neq 1$$

1. Use long division to find both the quotient and remainder when $2x^5 + 2x^4 - x^3 + 5x^2$ is divided by $x^4 - 3x^3$. (4 pts)

Solution:

We use long division to find the quotient and the remainder:

$$\begin{array}{r|l} & 2x + 8 \\ \hline x^4 - 3x^3 & 2x^5 + 2x^4 - x^3 + 5x^2 + 0x + 0 \\ & -(2x^5 - 6x^4) \\ \hline & 8x^4 - x^3 + 5x^2 + 0x + 0 \\ & -(8x^4 - 24x^3) \\ \hline & 23x^3 + 5x^2 \end{array}$$

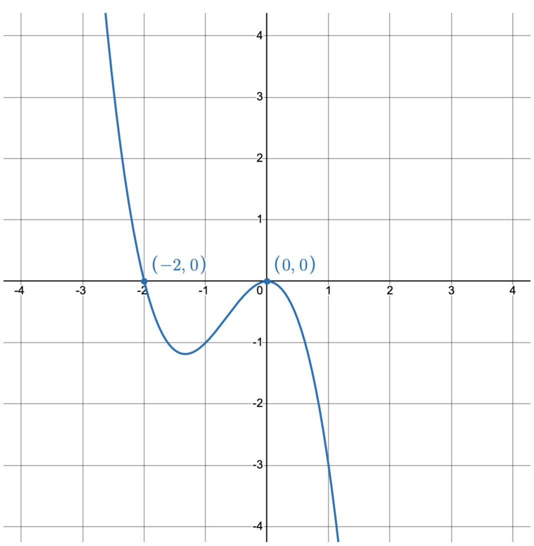
Thus the quotient is $\boxed{Q(x) = 2x + 8}$ and the remainder is $\boxed{R(x) = 23x^3 + 5x^2}$

2. Answer parts (a) and (b) for the polynomial function, $P(x)$, that satisfies **all** of the given information in i-iv.

   i. The graph has y-intercept $(0, 0)$.
   ii. The graph has end behavior consistent with $y = -x^3$
   iii. The graph crosses at x-intercept $(-2, 0)$
   iv. The graph has no other x-intercepts.

   (a) Sketch the shape of the graph of $P(x)$. **Label** the values of all intercepts on the graph. (4 pts)

   Solution:



(b) Write down a polynomial $P(x)$ that satisfies all of the given information. (2 pts)

Solution:

We start be recognizing that the x-intercepts of the graph correspond to factors of the polynomial. The polynomial with x-intercept at $x = 0$ has, by the factor theorem, factor $x - 0$ and the x-intercept at $x = -2$ corresponds to factor $x - (-2)$. An initial guess at the polynomial is $x(x + 2)$. However, we need the graph to cross at $x = -2$ (odd multiplicity) and bounce at $x = 0$ (even multiplicity). Our updated guess is $x^2(x + 2)$. Checking the end behavior, we see the graph of $y = x^2(x + 2)$ is not appropriate. One last modification leads us to a final answer: $P(x) = \boxed{-x^2(x + 2)}$.

3. Consider the following rational function: $f(x) = \dfrac{2x^4 + 10x^3 + 12x^2}{x^4 + 4x^3 - 5x^2}$. Answer the following: (12 pts)

(a) Find the x-coordinate of any hole(s) of $f(x)$. If there are none write NONE.

Solution:

First, we factor the numerator and the denominator of the rational function:

$$\begin{aligned} R(x) &= \frac{2x^4 + 10x^3 + 12x^2}{x^4 + 4x^3 - 5x^2} && (1) \\ &= \frac{2x^2\left(x^2 + 5x + 6\right)}{x^2\left(x^2 + 4x - 5\right)} && (2) \\ &= \frac{2x^2(x + 2)(x + 3)}{x^2(x + 5)(x - 1)} && (3) \end{aligned}$$

Hence this rational function reduces to $y = \dfrac{2(x + 2)(x + 3)}{(x + 5)(x - 1)}$ when $x \neq 0$. So there is a hole is located at $\boxed{x = 0}$.

(b) Find the y-coordinate of any hole(s) you found in part (a). If there are none write NONE.

Solution:

We plug in $x = 0$ into this simplified function to find the y-coordinate of the hole:
$y = \dfrac{2(0 + 2)(0 + 3))}{(0 + 5)(0 - 1)} = \boxed{-\dfrac{12}{5}}$

(c) Find all vertical asymptote(s) of $f(x)$. If there are none write NONE.

Solution:

Looking at the reduced function $y = \dfrac{2(x + 2)(x + 3)}{(x + 5)(x - 1)}$, there are vertical asymptotes when the denominator is zero. It is given by the vertical lines: $\boxed{x = -5}$ and $\boxed{x = 1}$

(d) Determine the end behavior of $f(x)$ and fill in the blanks: $f(x) \to$ ____ as $x \to -\infty$ and $f(x) \to$ ____ as $x \to \infty$.

Solution:

We notice that the Numerator and the Denominator of $R(x)$ both have degree 4. Hence as $x \to \pm\infty$ the function approaches the ratio of the leading terms, $\dfrac{2x^4}{x^4}$, which simplifies to 2 and thus has a Horizontal Asymptote given by $\boxed{y = 2}$. Filling in the arrow notation we get:

$$f(x) \to \boxed{2} \text{ as } x \to -\infty \text{ and } f(x) \to \boxed{2} \text{ as } x \to \infty$$

(e) Find all x-intercept(s) of $f(x)$. If there are none write NONE.

Solution:

In order to find x-intercept(s) of $f(x)$, we set the reduced function equal to zero.

$$\frac{2(x+2)(x+3)}{(x+5)(x-1)} = 0 \tag{4}$$

$$(x+2)(x-3) = 0 \tag{5}$$

$$x = -2, 3 \tag{6}$$

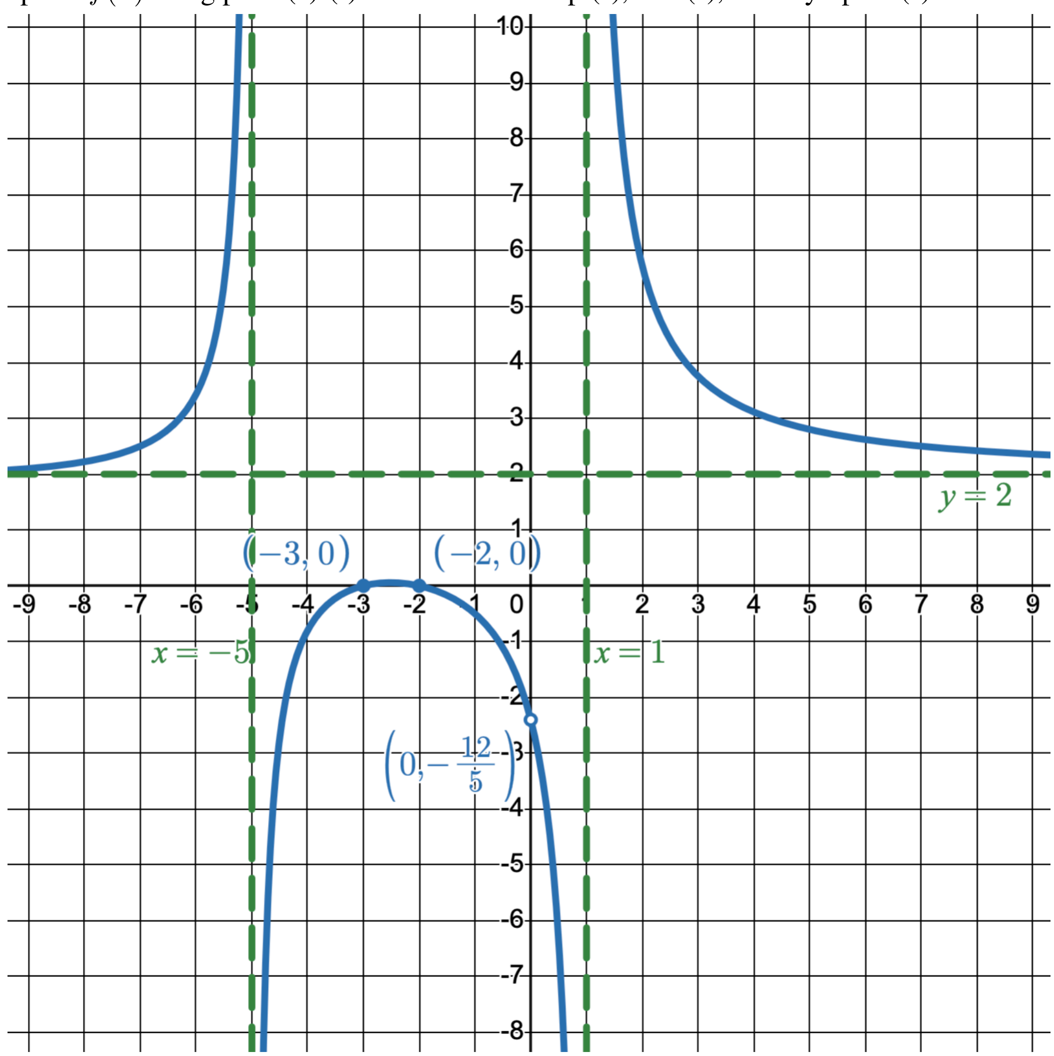
Hence the x-intercepts are $\boxed{(-3,0)}$ and $\boxed{(-2,0)}$

(f) Find the y-intercept. If there is none write NONE.

Solution:

$\boxed{\text{NONE}}$ since there is a hole at $x = 0$.

(g) Sketch the graph of $f(x)$ using parts (a)-(f). **Label** all intercept(s), hole(s), and asymptote(s) as relevant.

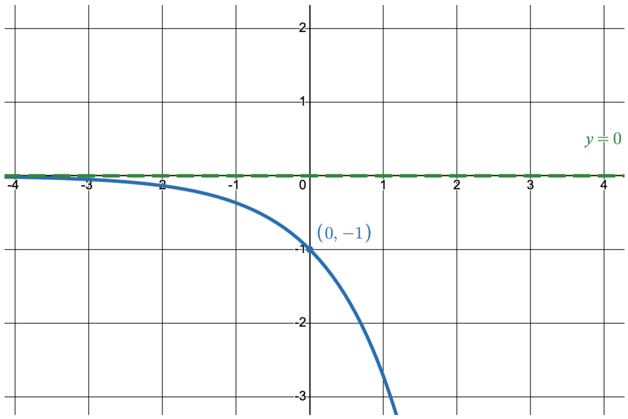


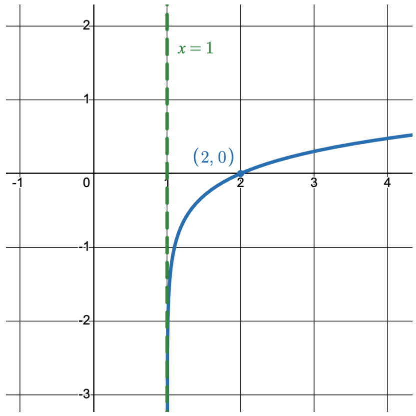
4. For parts (a) and (b) sketch the following graphs and be sure to label any asymptotes and intercepts for each graph:

(a) $g(x) = -e^x$ (3 pts)

(b) $h(x) = \log(x-1)$ (3 pts)

Solution:





(c) Use your graph in part (a) to fill in the blank: $g(x) \to$ ____ as $x \to -\infty$. (2 pts)

Solution:

$g(x) \to \boxed{0}$ as $x \to -\infty$

(d) What is the domain of $h(x) = \log(x-1)$, the same function from part (b) (2 pts)?

Solution:

The domain can be found from the graph or the domain is found by solving setting $x - 1 > 0$ thus getting $x > 1$: $\boxed{(1, \infty)}$

(e) For the function $h(x) = \log(x-1)$ given in part (b) find: $h\left(10^{2x}+1\right)$ (3 pts)

Solution:

$$h\left(10^{2x}+1\right) = \log(10^{2x}+1-1) \tag{7}$$
$$= \log(10^{2x}) \tag{8}$$
$$= \boxed{2x} \tag{9}$$

5. The following parts are unrelated.

(a) Simplify (rewrite without logs): $\log(1) - \log_2(32) - \ln\left(e^{-3}\right) + \log_3(9^x)$ (4 pts)

Solution:

$$\log(1) - \log_2(32) - \ln\left(e^{-3}\right) + \log_3(9^x) = 0 - \log_2(2^5) - (-3) + \log_3((3^2)^x) \tag{10}$$
$$= -5 + 3 + \log_3(3^{2x}) \tag{11}$$
$$= \boxed{-2 + 2x} \tag{12}$$

(b) Expand to rewrite as a sum/difference of logarithms without any exponents/roots, and simplify as usual: $\log_7\left(\frac{\sqrt{xy}}{7x}\right)$ (4 pts)

Solution:

$$\begin{aligned}
\log_7\left(\frac{\sqrt{xy}}{7x}\right) &= \log_7\left(\sqrt{xy}\right) - \log_7(7x) && (13)\\
&= \log_7\left(x^{\frac{1}{2}}y^{\frac{1}{2}}\right) - (\log_7(7) + \log_7(x)) && (14)\\
&= \log_7\left(x^{\frac{1}{2}}\right) + \log_7\left(y^{\frac{1}{2}}\right) - (1 + \log_7(x)) && (15)\\
&= \frac{1}{2}\log_7(x) + \frac{1}{2}\log_7(y) - 1 - \log_7(x) && (16)\\
&= \boxed{-\frac{1}{2}\log_7(x) + \frac{1}{2}\log_7(y) - 1} && (17)
\end{aligned}$$

6. Solve the following equations for x. If there are no solutions write "no solutions" (be sure to justify answer for full credit).

(a) $4^{5-7x} = 16$ (4pts)

(b) $3^{2x-1} = 2^x$ (4 pts)

Solution:

Using one-to-one property of exponential/logarithmic functions

$$\begin{aligned}
4^{5-7x} &= 16 & 3^{2x-1} &= 2^x && (18)\\
4^{5-7x} &= 4^2 & \log_3(3^{2x-1}) &= \log_3(2^x) && (19)\\
5 - 7x &= 2 & 2x - 1 &= x\log_3(2) && (20)\\
-7x &= 2 - 5 & 2x - x\log_3(2) &= 1 && (21)\\
-7x &= -3 & x(2 - \log_3(2)) &= 1 && (22)\\
x &= \boxed{\frac{3}{7}} & x &= \boxed{\frac{1}{2 - \log_3(2)}} && (23)
\end{aligned}$$

(c) $\ln\left(10 + x^2\right) = \ln 7 + \ln(x)$ (4pts)

(d) $2\log_2\left(3x^{1/2}\right) = 6$ (4 pts)

Solution:

$$\begin{aligned}
\ln^{(10+x^2)} &= \ln 7 + \ln(x) & 2\log_2\left(3x^{1/2}\right) &= 6 && (24)\\
\ln(10 + x^2) &= \ln(7x) & \log_2\left(3x^{1/2}\right) &= 3 && (25)\\
10 + x^2 &= 7x & 2^3 &= 3x^{1/2} && (26)\\
x^2 - 7x + 10 &= 0 & \frac{8}{3} &= x^{1/2} && (27)\\
(x-5)(x-2) &= 0 & x &= \boxed{\frac{64}{9}} && (28)\\
x &= \boxed{5, 2} & & && (29)
\end{aligned}$$

7. 340 Siberian cats are introduced onto an island and scientists estimate that the cat population will double every three years. Answer the following questions: (9 pts)

   (a) Find a model of the form $P(t) = P_\circ 2^{t/a}$ that models the population size of cats.

   Solution:

   Here $P_\circ = 340$ and $a = 3$,

$$\boxed{P(t) = 340 \cdot 2^{t/3}}$$

   (b) How many cats are expected to be on the island in 9 years? Leave your answer in exact form (do not attempt to approximate as a decimal value).

   Solution:

   Since $P(9) = 340 \cdot 2^{9/3} = 340 \cdot 2^3 = 8 \cdot 340 = 2720$, $\boxed{\text{P(9) = 2720}}$.

   (c) According to the model found in part (a), how long until the population reaches 2000 cats? Leave your answer in exact form (do not attempt to approximate as a decimal value).

   Solution:

   We want to solve for t when $P(t) = 2000$.

$$P(t) = 2000 \tag{30}$$
$$340 \cdot 2^{t/3} = 2000 \tag{31}$$
$$2^{t/3} = \frac{100}{17} \tag{32}$$
$$\log_2(2^{t/3}) = \log_2\left(\frac{100}{17}\right) \tag{33}$$
$$\frac{t}{3} = \log_2\left(\frac{100}{17}\right) \tag{34}$$
$$t = 3\log_2\left(\frac{100}{17}\right) \tag{35}$$

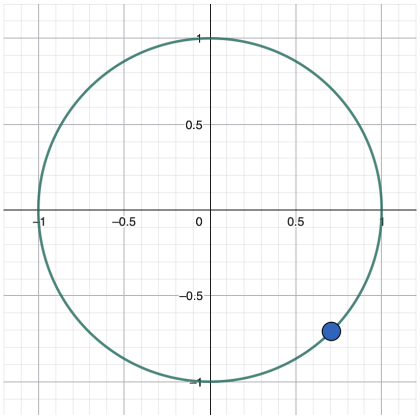
   When $\boxed{t = 3\log_2\left(\frac{100}{17}\right) \text{ years}}$, $P(t) = 2000$.

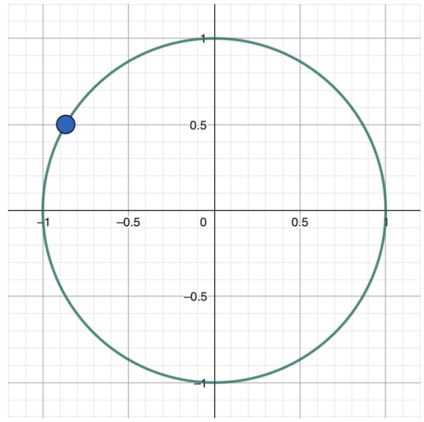
8. Plot the terminal point for each real number on the unit circle (you do not need to find the coordinates of the point).

(a) $t = \frac{7\pi}{4}$ (2 pts)

(b) $t = -\frac{7\pi}{6}$ (2 pts)

Solution:





9. Let $\left(\frac{1}{4}, y\right)$ be the terminal point on the unit circle corresponding to real number t. Suppose we also know $\sin t < 0$. Use this information to answer the following:

(a) Considering all given information, what quadrant does the terminal point of t lie in? (2 pts)

Solution:

Since $\cos t = \frac{1}{4} > 0$ and $\sin t < 0$ the terminal point is in $\boxed{\text{quadrant IV}}$

(b) Find the value for y. (3 pts)

Solution:

Since the point is on the unit circle, we can solve for y in the equation:

$$\left(\frac{1}{4}\right)^2 + y^2 = 1 \tag{36}$$

$$\frac{1}{16} + y^2 = \frac{16}{16} \tag{37}$$

$$y^2 = \frac{15}{16} \tag{38}$$

$$y = \boxed{-\frac{\sqrt{15}}{4}} \tag{39}$$

where we've taken the minus sign since $\sin t < 0$.

(c) Find $\cos t$ (2 pts)

Solution:

The cosine of t is the x coordinate of the point on the unit circle:

$$\boxed{\cos t = \frac{1}{4}} \tag{40}$$

(d) Find $\sec t$ (2 pts)

Solution:

Secant is the reciprocal of cosine:

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{1}{4}} = \boxed{4} \tag{41}$$

10. Find the exact value of each of the following. If a value does not exist write DNE. (15 pts)

(a) $\sin\left(\frac{\pi}{3}\right)$

(b) $\cos\left(\frac{3\pi}{4}\right)$

Solution:

$$\sin\left(\frac{\pi}{3}\right) = \boxed{\frac{\sqrt{3}}{2}}$$

$$\cos\left(\frac{3\pi}{4}\right) = \boxed{-\frac{\sqrt{2}}{2}}$$

(c) $\tan\left(-\frac{2\pi}{3}\right)$

(d) $\sin(0)$

Solution:

$$\tan\left(-\frac{2\pi}{3}\right) = \frac{\sin\left(-\frac{2\pi}{3}\right)}{\cos\left(-\frac{2\pi}{3}\right)} = \frac{\left(-\frac{\sqrt{3}}{2}\right)}{\left(-\frac{1}{2}\right)} = \boxed{\sqrt{3}}$$

$$\sin(0) = \boxed{0}$$

(e) $\csc\left(\frac{5\pi}{6}\right)$

Solution:

$$\csc\left(\frac{5\pi}{6}\right) = \frac{1}{\sin\left(\frac{5\pi}{6}\right)} = \frac{1}{\left(\frac{1}{2}\right)} = \boxed{2}$$

11. Simplify the expression: $e^x\left(e^x + e^{2x}\right) - 2\left(e^x\right)^3$ (4 pts)

Solution:

$$e^x\left(e^x + e^{2x}\right) - 2\left(e^x\right)^3 = e^{2x} + e^{3x} - 2e^{3x} \tag{42}$$

$$= \boxed{e^{2x} - e^{3x}} \tag{43}$$