1. (30 pts) The following three problems are <u>not related</u>.

   (a) Beulah Bee travels along the path $\mathbf{r}(t) = (6t)\mathbf{i} + (8\cos t)\mathbf{j} + (8\sin t)\mathbf{k}$, starting at $t = 0$, and stops after covering a distance of 15π units. What are the coordinates of the bee's position now? Simplify your answer.

   (b) The temperature at a point (x, y) is $T(x, y)$ degrees. A snail crawls so that its position after t minutes is
$$x = -2 + \frac{6}{\sqrt{1+t}}, \quad y = 7 + \sqrt{1+t}.$$
   Suppose $T_x(1, 9) = 3$ and $T_y(1, 9) = 2$. How fast is the temperature changing on the snail's path after 3 minutes?

   (c) Show that $\displaystyle\lim_{(x,y)\to(5,5)} \frac{(x-5)(y-5)}{(x-5)^2 + (y-5)^2}$ does not exist.

Solution:

(a)
$$\begin{aligned}
\mathbf{r}(t) &= \langle 6t, 8\cos t, 8\sin t\rangle \\
\mathbf{r}'(t) &= \langle 6, -8\sin t, 8\cos t\rangle \\
\left|\mathbf{r}'(t)\right| &= \sqrt{6^2 + 64\sin^2 t + 64\cos^2 t} \\
&= \sqrt{36 + 64} = \sqrt{100} = 10
\end{aligned}$$

Beulah travels from $t = 0$ to $t = a$, covering a distance of
$$\begin{aligned}
15\pi &= \int_0^a 10\,dt = 10t\Big]_0^a \\
&= 10a \implies a = \frac{15\pi}{10} = \frac{3\pi}{2}.
\end{aligned}$$

Beulah is now at $\mathbf{r}\left(\frac{3\pi}{2}\right)$ which corresponds to the point $\left(6 \cdot \frac{3\pi}{2}, 8\cos\left(\frac{3\pi}{2}\right), 8\sin\left(\frac{3\pi}{2}\right)\right) = \boxed{(9\pi, 0, -8)}$.

(b) First calculate $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.
$$\frac{dx}{dt} = -\frac{3}{(1+t)^{3/2}} \quad \text{and} \quad \frac{dy}{dt} = \frac{1}{2\sqrt{1+t}}.$$

At $t = 3$,
$$\frac{dx}{dt} = -\frac{3}{2^3} = -\frac{3}{8} \quad \text{and} \quad \frac{dy}{dt} = \frac{1}{2\cdot 2} = \frac{1}{4}.$$

The change in temperature when $t = 3$, $x = 1$, and $y = 9$ is
$$\begin{aligned}
\frac{dT}{dt} &= \frac{\partial T}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial T}{\partial y}\cdot\frac{dy}{dt} \\
&= T_x(1,9)\left(-\frac{3}{8}\right) + T_y(1,9)\cdot\frac{1}{4} \\
&= 3\left(-\frac{3}{8}\right) + 2\cdot\frac{1}{4} = \boxed{-\frac{5}{8}^\circ\text{/min}}.
\end{aligned}$$

(c) Let $f(x,y) = \dfrac{(x-5)(y-5)}{(x-5)^2+(y-5)^2}$.

First approach $(5,5)$ along the line $y = 5$. Then $f(x,5) = \dfrac{0}{(x-5)^2} = 0$ for $x \neq 5$, so $f(x,y) \to 0$. By symmetry, $f(x,y) \to 0$ approaching $(5,5)$ along the line $x = 5$.

Approaching $(5,5)$ along the line $y = x$, $f(x,x) = \dfrac{(x-5)^2}{2(x-5)^2} = \dfrac{1}{2}$, so along this line $f(x,y) \to \dfrac{1}{2}$. Therefore the limit does not exist.

2. (36 pts) Consider the surface $z = f(x,y) = (x-2)y^2$ with point P at $(4,-1,2)$.

(a) Find an equation for the plane tangent to the surface at P.

(b) Find a linear approximation for $f(x,y)$ centered at $(4,-1)$ and use it to estimate the value of $f(4.05,-1.1)$.

(c) Use Taylor's Formula to find an upper bound for the error in the linear approximation of $f(x,y)$ when $3.8 \leq x \leq 4.2$ and $-1.1 \leq y \leq -0.9$.

(d) Sketch the level curve $z = 2$ and the vector $\nabla f(4,-1)$ on the axes below.

Solution:

(a) A plane tangent to $z = f(x,y)$ has the equation

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

For this function

$$f_x = y^2 \qquad f_x(4,-1) = 1$$
$$f_y = 2(x-2)y \qquad f_y(4,-1) = -4.$$

An equation of the tangent plane is therefore

$$\boxed{z - 2 = (x-4) - 4(y+1)}$$
$$z = x - 4y - 6.$$

(b) A linear approximation corresponding to the tangent plane is

$$L(x,y) = 2 + (x-4) - 4(y+1).$$

Then

$$f(4.05,-1.1) \approx L(4.05,-1.1) = 2 + 0.05 - 4(-0.1) = \boxed{2.45}.$$

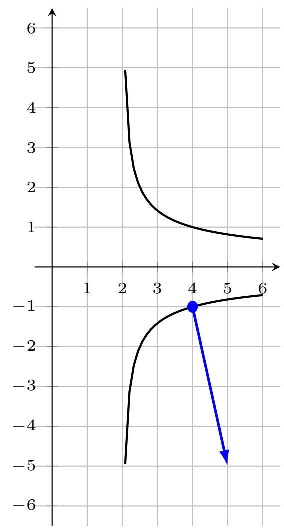
(c) For $3.8 \leq x \leq 4.2$ and $-1.1 \leq y \leq -0.9$,

$$f_{xx} = 0$$
$$f_{yy} = 2(x-2) \qquad |f_{yy}| \leq 2(4.2-2) = 4.4$$
$$f_{xy} = 2y \qquad |f_{xy}| \leq 2.2$$

Let the upper bound $M = 4.4$. Then

$$\begin{aligned} |E(x,y)| &\le \frac{1}{2!}M\left(|x-4|+|y+1|\right)^2 \\ &\le \frac{4.4}{2}(0.2+0.1)^2 \\ &= 2.2(0.09) = \boxed{0.198}. \end{aligned}$$

(d) Note that $\nabla f(4,-1) = \langle f_x(4,-1), f_y(4,-1)\rangle = \langle 1,-4\rangle$. The gradient vector at $(4,-1)$ is shown below, along with the graph of $(x-2)y^2 = 2$ which can be written as $x = \dfrac{2}{y^2}+2$ (similar to HW 1 #2b).



3. (34 pts) Suppose the elevation of the land near Chet Chipmunk's home is given by

$$g(x,y) = \frac{x^3}{3} - \frac{y^2}{2} + 2xy + 2$$

where x and y are measured in meters.

(a) Find the critical points (x,y) where the land has local extrema or saddle points. Use the Second Derivatives Test to classify the points.

(b) A nearby trail runs along the line $y = x$ for $-5 \le x \le 3$. At what x-coordinate does the trail have a local maximum?

(c) Chet is at $Q(2,0)$ when he spots a fox at $R(0,1)$.

i. Find the directional derivative of g at Q in the direction of $\overrightarrow{QR}$.

ii. Chet decides to scramble to higher ground as quickly as possible. In which direction should he move? Express your simplified answer in terms of a unit vector.

Solution:

(a) Begin by calculating the first partial derivatives:

$$g_x = x^2 + 2y \qquad g_y = -y + 2x.$$

If $g_y = -y + 2x = 0$, then $y = 2x$. Substituting into $g_x = 0$ gives $x^2 + 4x = 0$, so $x = 0$ or -4. The critical points are $\boxed{(0,0) \text{ and } (-4,-8)}$. Next calculate the second partial derivatives:

$$\begin{aligned} g_{xx} &= 2x & g_{xx}(0,0) &= 0 & g_{xx}(-4,-8) &= -8 \\ g_{yy} &= -1 & g_{yy}(0,0) &= -1 & g_{yy}(-4,-8) &= -1 \\ g_{xy} &= 2 & g_{xy}(0,0) &= 2 & g_{xy}(-4,-8) &= 2. \end{aligned}$$

Applying the Second Derivatives Test to the point $(0,0)$ gives

$$D = g_{xx}g_{yy} - (g_{xy})^2 = 0 - 2^2 = -4 < 0,$$

so there is a $\boxed{\text{saddle point at } (0,0)}$. For the point $(-4,-8)$,

$$D = g_{xx}g_{yy} - (g_{xy})^2 = 8 - 2^2 = 4 > 0 \quad \text{and} \quad fxx(-4,-8) < 0,$$

so there is a $\boxed{\text{local maximum at } (-4,-8)}$.

(b) The plane $y = x$ meets the surface at the curve

$$g(x,x) = \frac{x^3}{3} - \frac{x^2}{2} + 2x^2 + 2 = \frac{x^3}{3} + \frac{3x^2}{2} + 2.$$

Given

$$g'(x,x) = x^2 + 3x \quad \text{and} \quad g''(x,x) = 2x + 3,$$

the curve has critical points at $x = 0, -3$ where $g' = 0$. There is a local minimum at $x = 0$ where $g''(0,0) > 0$ and a local maximum at $\boxed{x = -3}$ where $g''(-3,-3) < 0$.

(c) i. Let $\mathbf{u} = \dfrac{\mathbf{QR}}{|\mathbf{QR}|} = \dfrac{\langle -2,1\rangle}{\sqrt{2^2+1^2}} = \dfrac{\langle -2,1\rangle}{\sqrt{5}}$. Then the directional derivative is

$$\begin{aligned} D_{\mathbf{u}}g(2,0) &= \nabla g(2,0) \cdot \frac{\langle -2,1\rangle}{\sqrt{5}} \\ &= \langle g_x(2,0), g_y(2,0)\rangle \cdot \frac{\langle -2,1\rangle}{\sqrt{5}} \\ &= \langle 4,4\rangle \cdot \frac{\langle -2,1\rangle}{\sqrt{5}} \\ &= \frac{4(-2) + 4 \cdot 1}{\sqrt{5}} = \boxed{-\frac{4}{\sqrt{5}}}. \end{aligned}$$

ii. Chet should move in the direction of the gradient vector

$$\frac{\nabla g(2,0)}{|\nabla g(2,0)|} = \frac{\langle 4,4\rangle}{|\langle 4,4\rangle|} = \frac{4\,\langle 1,1\rangle}{4\,|\langle 1,1\rangle|} = \boxed{\frac{\langle 1,1\rangle}{\sqrt{2}}} = \boxed{\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle}.$$