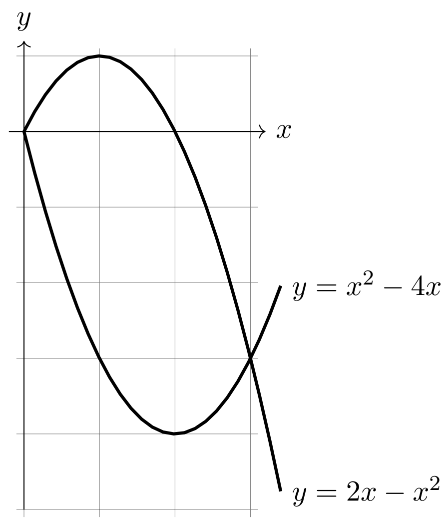
1. (34 points) Consider the region $\mathcal{R}$ between the two curves $y = x^2 - 4x$ and $y = 2x - x^2$ shown here:



(a) Find the area of the region $\mathcal{R}$.

(b) Find the x-coordinate of the centroid of this region.

(c) Now, consider a new solid generated by rotating the region $\mathcal{R}$ around the line $y = 2$. Set up, but do not evaluate, an integral to calculate the volume of the solid.

(d) Finally, consider the solid generated by rotating the region $\mathcal{R}$ around the line $x = -1$. Set up, but do not evaluate, an integral to calculate the volume of the solid.

Solution:

(a) To find the area, we first find where the curves intersect by setting them equal to each other, and solving for x:

$$2x - x^2 = x^2 - 4x \implies 6x = 2x^2 \implies 0 = x^2 - 3x \implies 0 = x(x-3),$$

so the curves solve at $x = 0$, $x = 3$. Then the area is given by

$$\begin{aligned}
\int_0^3 \left[(2x - x^2) - (x^2 + 4x)\right]\, dx &= \int_0^3 (6x - 2x^2)\, dx \\
&= \left(3x^2 - \frac{2}{3}x^3\right)\Big|_0^3 \\
&= 27 - \frac{2}{3}\cdot 27 \\
&= 9.
\end{aligned}$$

(b) To find the centroid of the region, we use the area value from the previous part:

$$\begin{aligned}
\bar{x} &= \frac{1}{9}\int_0^3 x\left(2x - x^2 - x^2 + 4x\right)\, dx \\
&= \frac{1}{9}\int_0^3 \left(6x - 2x^3\right)\, dx \\
&= \frac{1}{9}\left(2x^3 - \frac{1}{2}x^4\right)\Big|_0^3 \\
&= \frac{1}{9}\left(54 - \frac{81}{2}\right)
\end{aligned}$$

$$= \frac{3}{2}.$$

(c) This will be a washer-method integral:

$$\text{Volume} = \int_0^3 \left[\pi(2 - x^2 + 4x)^2 - \pi(2 - 2x + x^2)^2\right]\, dx.$$

(d) This will be a shell-method integral:

$$\begin{aligned}\text{Volume} &= \int_0^3 2\pi(x+1)\left(2x - x^2 - x^2 + 4x\right)\, dx \\ &= \int_0^3 2\pi(x+1)\left(6x - 2x^2\right)\, dx\end{aligned}$$

2. (12 points) Solve the differential equation $\dfrac{dS}{dr} + \dfrac{2}{r}S = 0$ with initial value $S(2) = 3$. Assume $r > 0$ and $S > 0$. Simplify and express your answer in the form $S = S(r)$.

Solution:

$$\begin{aligned}\frac{dS}{dr} + \frac{2}{r}S &= 0 \\ \frac{dS}{dr} &= -\frac{2}{r}S \\ \int \frac{dS}{S} &= \int -\frac{2}{r}\, dr \\ \ln S &= -2\ln r + C \quad \text{(since } r, S > 0) \\ S &= e^{-2\ln r + C} = Ae^{-2\ln r} = Ar^{-2}\end{aligned}$$

Use the initial value $S(2) = 3$ to find that $3 = A(2)^{-2} \Rightarrow A = 12$.

$$\boxed{S = 12r^{-2}}$$

3. (24 points) The following problems are not related.

(a) Find the length of the curve $g(y) = \frac{1}{3}y^3 + \frac{1}{4y}$ from $y = 1$ to $y = 2$.

(b) Suppose you have a hemispherical bowl of radius 2 feet that is filled to a depth of 1 foot with water. What is the surface area of the bowl that is in contact with the water? Set up the integral but you need not evaluate it.

Solution:

(a) To compute the length of the curve, we need

$$g'(y) = y^2 - \frac{1}{4y^2}$$

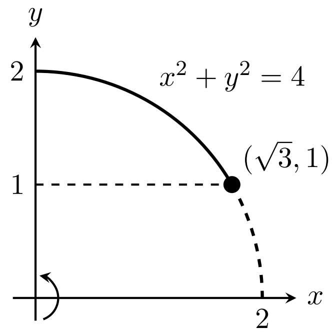
and

$$ds = \sqrt{1 + (g'(y))^2}\, dy = \sqrt{1 + \left(y^2 - \frac{1}{4y^2}\right)^2}\, dy = \sqrt{y^4 + \frac{1}{2} + \frac{1}{16y^4}}\, dy = y^2 + \frac{1}{4y^2}\, dy.$$

Then, the length is given by

$$L = \int_a^b ds = \int_1^2 y^2 + \frac{1}{4y^2}\, dy = \frac{y^3}{3} - \frac{1}{4y}\Big|_1^2 = \frac{2^3}{3} - \frac{1}{4\cdot 2} - \frac{1}{3} + \frac{1}{4} = \boxed{\frac{59}{24}.}$$

(b) To obtain the surface area, we can rotate the equation of a circle about the y-axis. Visually, we have



For a radius of 2, the circle is given by

$$x^2 + y^2 = 2^2 \implies y = \sqrt{4 - x^2}.$$

Then,

$$y' = \frac{-2x}{2\sqrt{4-x^2}} = -\frac{x}{\sqrt{4-x^2}}$$

and

$$ds = \sqrt{1 + (y')^2}\, dx = \sqrt{1 + \left(-\frac{x}{\sqrt{4-x^2}}\right)^2}\, dx = \frac{2}{\sqrt{4-x^2}}\, dx.$$

The radius from the y-axis is $r = x$. Putting all of this into our surface area formula yields

$$SA = \int_a^b 2\pi r\, ds = \boxed{\int_0^{\sqrt{3}} 4\pi \frac{x}{\sqrt{4-x^2}}\, dx.}$$

Alternatively, with respect to y, we have that

$$x = \sqrt{4 - y^2}$$

meaning the radius can now be written $r = x = \sqrt{4 - y^2}$. In terms of y, ds is given by

$$ds = \sqrt{1 + (x')^2}\, dy = \frac{2}{\sqrt{4-y^2}}\, dy.$$

Then, the surface area in terms of y can be computed as

$$SA = \int_a^b 2\pi r\, ds = \boxed{\int_1^2 2\pi\sqrt{4-y^2}\frac{2}{\sqrt{4-y^2}}\, dy = 4\pi \int_1^2 dy = 4\pi.}$$

4. (18 points) Determine whether the sequences below converge or diverge. If a sequence converges, find the limit. Fully justify your work, and state any theorems that you use. For each sequence, you may assume that $n = 1, 2, 3, \ldots$

(a) $a_n = \dfrac{\sin(n)}{n^{1/2}}$

(b) $b_n = \dfrac{(2n+1)!}{(2n-1)!}$

(c) $c_n = \ln(6n^2 + 7) - \ln(n^2 + 4)$

Solution:

(a) Since $-1 \leq \sin(n) \leq 1$, we have that

$$-\frac{1}{n^{1/2}} \leq \frac{\sin(n)}{n^{1/2}} \leq \frac{1}{n^{1/2}}.$$

Taking the limit as $n \to \infty$ of this ineqaultiy yields

$$0 \leq \lim_{n\to\infty} \frac{\sin(n)}{n^{1/2}} \leq 0.$$

Then, by the squeeze theorem, the sequence $\boxed{\text{converges with } \lim_{n\to\infty} a_n = 0.}$

(b) We can simplify the sequence as

$$b_n = \frac{(2n+1)!}{(2n-1)!} = \frac{(2n+1)(2n)!}{(2n-1)!} = \frac{(2n+1)(2n)(2n-1)!}{(2n-1)!} = (2n+1)(2n).$$

Now, taking limits we find that $\boxed{\lim_{n\to\infty} b_n = \infty;\ \text{diverges.}}$

(c) The way c_n is written, the limit yields the indeterminate form $\lim_{n\to\infty} c_n = \infty - \infty$. However, with a little algebra, we find that

$$\begin{aligned}
\lim_{n\to\infty} c_n &= \lim_{n\to\infty} \ln(6n^2+7) - \ln(n^2+4) \\
&= \lim_{n\to\infty} \ln\left(\frac{6n^2+7}{n^2+4}\right) \\
&= \ln\left(\lim_{n\to\infty} \frac{6n^2+7}{n^2+4}\right) \\
&\overset{LH}{=} \ln\left(\lim_{n\to\infty} \frac{12n}{2n}\right) \\
&= \ln(6).
\end{aligned}$$

In other words, the sequence $\boxed{\text{converges with } c_n \to \ln(6) \text{ as } n \to \infty.}$

5. (12 points) Consider the sequence $a_n = \left(-\frac{3}{4}\right)^n$ for $n = 1, 2, 3, \ldots$

(a) Find the value of s_2, the second partial sum.

(b) Determine if the series $\sum_{n=1}^{\infty} a_n$ converges or diverges. If it converges, find its limit.

(c) Determine whether $\sum_{n=1}^{\infty} (a_n - 1)$ converges or diverges. If it converges, find its limit.

Solution:

(a) $s_2 = a_1 + a_2 = -\frac{3}{4} + \left(-\frac{3}{4}\right)^2 = \boxed{-\frac{3}{16}.}$

(b) Since we can write

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^n = \sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)\left(-\frac{3}{4}\right)^{n-1}$$

$\sum_{n=1}^{\infty} a_n$ is just a geometric series with $a = -\frac{3}{4}$ and $r = -\frac{3}{4}$. Since $|r| < 1$, the series converges and

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^n = \frac{a}{1-r} = \frac{-3/4}{1-(-3/4)} = \boxed{-\frac{3}{7}.}$$

(c) Since $\sum_{n=1}^{\infty} a_n$ converges, we know that $\lim_{n\to\infty} a_n = 0$. So,

$$\lim_{n\to\infty} (a_n - 1) = 0 - 1 = -1 \neq 0.$$

By the divergence test, $\sum_{n=1}^{\infty}(a_n - 1)$ must $\boxed{\text{diverge.}}$