1. [2350/101525 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

   (a) The function $f(x,y) = x^2 + \sqrt[3]{y}$ has no critical points.

   (b) The level curves of the function $f(x,y) = \dfrac{1}{2 - 2x + x^2 + y^2}$ are circles or points.

   (c) If $f(x,y) \to 5$ as $(x,y) \to (0,0)$ along the line $y = x$ and $f(x,y) \to 5$ as $(x,y) \to (0,0)$ along the parabola $y = x^2$, then $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ must equal 5.

   (d) There exists a real number c such that $f(x,y) = \begin{cases} \dfrac{|x|}{|x| + |y|} & (x,y) \neq (0,0) \\ c & (x,y) = (0,0) \end{cases}$ is continuous throughout $\mathbb{R}^2$.

   (e) The linearization of $g(x,y) = \dfrac{2x+3}{4y+1}$ at the origin is $L(x,y) = 3 + 2x - 12y$.

   SOLUTION:

   (a) **FALSE**

   $$f_x(x,y) = 2x \qquad f_y(x,y) = \frac{1}{y^{2/3}}$$

   Since $f_x(0,0) = 0$ and $f_y(0,0)$ fails to exist, $(0,0)$ is a critical point.

   (b) **TRUE** $\dfrac{1}{2 - 2x + x^2 + y^2} = k \implies 2 - 2x + x^2 + y^2 = \dfrac{1}{k} \implies (x-1)^2 + y^2 = \dfrac{1}{k} - 1$ which is a circle unless $k = 1$ in which case it is a point.

   (c) **FALSE** The limit may not exist. $f(x,y)$ must approach 5 along *every* path to the origin.

   (d) **FALSE** Approaching the origin along the x-axis $f(x,y) \to 1$ and approaching along the y-axis $f(x,y) \to 0$. Therefore the limit at the origin fails to exist so regardless of the choice of c, $f(x,y)$ cannot be continuous at the origin and thus cannot be continuous throughout $\mathbb{R}^2$.

   (e) **TRUE**

   $$g_x = \frac{2}{4y+1} \implies g_x(0,0) = 2$$

   $$g_y = -\frac{4(2x+3)}{(4y+1)^2} \implies g_y(0,0) = -12$$

   $$g(0,0) = 3 \implies L(x,y) = 3 + 2x - 12y$$

   ■

2. [2350/101525 (19 pts)] Buzz and Emperor Zurg are still fighting. Zurg's energy shield is in the shape of $x^2 + 2y^2 + z^2 = 20$ and he also deployed a magnetic field having a strength of $M(x,y,z) = x + y^2 + 2z$ to try to contain Buzz.

   (a) (7 pts) Star Command wants to build a landing platform in the shape of a plane tangent to the energy shield at the point $(1, -3, 1)$. Find the equation of the landing platform, writing your answer in the form $ax + by + cz = d$.

   (b) (12 pts) Buzz has been tasked to find the point(s) on the energy shield where the strength of the magnetic field is a minimum. Find this/these point(s) and the strength of the magnetic field there.

   SOLUTION:

(a) The energy shield is a level surface of the function $g(x,y,z) = x^2 + 2y^2 + z^2$.

$$\nabla g(x,y,z) = \langle 2x, 4y, 2z\rangle \implies \nabla g(1,-3,1) = \langle 2, -12, 2\rangle$$

$$2(x-1) - 12(y+3) + 2(z-1) = 0 \implies 2x - 12y + 2z = 40 \implies x - 6y + z = 20$$

Alternatively, we have $z = f(x,y) = \sqrt{20 - x^2 - 2y^2}$ where we use the positive square root since that contains the point in question. Then the tangent plane can be obtained from

$$z = f(1,-3) + f_x(1,-3)(x-1) + f_y(1,-3)(y+3)$$

$$= 1 + \left.\frac{-x}{\sqrt{20 - x^2 - 2y^2}}\right|_{(1,-3)} (x-1) + \left.\frac{-2y}{\sqrt{2 - x^2 - 2y^2}}\right|_{(1,-3)} (y+3)$$

$$= 1 - (x-1) + 6(y+3) = 1 - x + 1 + 6y + 18 \implies x - 6y + z = 20$$

(b) Use Lagrange multipliers with the function to be minimized (the objective function) being the magnetic field and the constraint being the energy shield. The gradient of the constraint was computed in part (a) and to that we add $\nabla M(x,y,z) = \langle 1, 2y, 2\rangle$. Then

$$1 = \lambda(2x) \tag{1}$$

$$2y = \lambda(4y) \tag{2}$$

$$2 = \lambda(2z) \tag{3}$$

Note that neither x nor z can be zero. Thus (1) implies $\lambda = 1/2x$ and (3) implies $\lambda = 1/z$. Together these give $z = 2x$. If $y = 0$ in (2), the constraint yields $x^2 + (2x)^2 = 20 \implies x = \pm 2 \implies z = \pm 4$ giving critical points of $(2,0,4)$ and $(-2,0,-4)$. If $\lambda = 1/2$ in (2), then from (1) $x = 1$ and from (3) $z = 2$. Using these in the constraint we have $1^2 + 2y^2 + 2^2 = 20 \implies y = \pm\sqrt{15/2}$ giving critical points $\left(1, \sqrt{\frac{15}{2}}, 2\right)$ and $\left(1, -\sqrt{\frac{15}{2}}, 2\right)$. Finally

$$M(2,0,4) = 2 + 0^2 + 2(4) = 10$$

$$M(-2,0,-4) = -2 + 0^2 + 2(-4) = -10$$

$$M\left(1, \pm\sqrt{\tfrac{15}{2}}, 2\right) = 1 + \frac{15}{2} + 2(2) = \frac{25}{2}$$

The minimum strength in the magnetic field is -10 at $(-2,0,-4)$. Buzz's task is accomplished.

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3. [2350/101525 (25 pts)] Our hummingbird has decided to visit the exam again. She is flying around in your neighbor's yard where the intensity of sweet-smelling nectar, N, is given by $N(x,y,z) = xy^2 + x^2z + yz^2$.

(a) (7 pts) If she is at the point $P(1,-1,2)$, find the directional derivative of $N(x,y,z)$ in the direction from P to $Q(2,-1,3)$.

(b) (4 pts) If she is at the point $R(1,1,1)$, is there a direction in which she can fly such that the rate of change of the nectar intensity is -25? If there is, find it. If not, explain why not.

(c) (14 pts) She is now flying along the path $\mathbf{r}(t) = \langle \sin 3t, e^{2t}, \sin 2t\rangle$.

i. (7 pts) When she is at the point $B(-1, e^{\pi}, 0)$ what is the rate of change of the nectar intensity with respect to time?

ii. (7 pts) How fast is the nectar intensity changing with respect to distance or arc length at the point B?

SOLUTION:

(a)

$$\overrightarrow{PQ} = \langle 2-1, -1-(-1), 3-2\rangle = \langle 1,0,1\rangle \implies \mathbf{u} = \frac{1}{\sqrt{2}}\langle 1,0,1\rangle$$

$$\nabla N(x,y,z) = \langle y^2 + 2xz, 2xy + z^2, x^2 + 2yz\rangle$$

$$D_{\mathbf{u}}N(1,-1,2) = \nabla N(1,-1,2)\cdot\mathbf{u} = \langle (-1)^2 + 2(1)(2), 2(1)(-1) + 2^2, 1^2 + 2(-1)(2)\rangle \cdot \left(\frac{1}{\sqrt{2}}\langle 1,0,1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\langle 5, 2, -3\rangle \cdot \langle 1,0,1\rangle = \frac{2}{\sqrt{2}} = \sqrt{2}$$

(b) No.

$$\nabla N(1,1,1) = \left\langle (1)^2 + 2(1)(1), 2(1)(1) + 1^2, 1^2 + 2(1)(1) \right\rangle = \langle 3,3,3\rangle \implies \|\nabla N(1,1,1)\| = 3\sqrt{3}$$

The minimum rate of change of the nectar intensity at R is $-\|\nabla N(1,1,1)\| = -3\sqrt{3}$ which is greater than -25.

(c) i. The hummingbird is at point B when $t = \pi/2$ and we have $\mathbf{r}'(t) = \left\langle 3\cos 3t, 2e^{2t}, 2\cos 2t \right\rangle$.

$$\left.\frac{\mathrm{d}N}{\mathrm{d}t}\right|_{t=\pi/2} = \nabla N(-1, e^{\pi}, 0)\cdot \mathbf{r}'\left(\tfrac{\pi}{2}\right) = \left\langle e^{2\pi}, -2e^{\pi}, 1\right\rangle \cdot \langle 0, 2e^{\pi}, -2\rangle = -4e^{2\pi} - 2 = -2\left(2e^{2\pi}+1\right)$$

ii.

$$\frac{\mathrm{d}N}{\mathrm{d}s} = \frac{\mathrm{d}N}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\mathrm{d}N}{\mathrm{d}t}\left(\frac{1}{\|\mathbf{r}'\|}\right) = \frac{-2\left(2e^{2\pi}+1\right)}{\|\langle 0, 2e^{\pi}, -2\rangle\|} = \frac{-2\left(2e^{2\pi}+1\right)}{\sqrt{4e^{2\pi}+4}} = -\frac{2e^{2\pi}+1}{\sqrt{e^{2\pi}+1}}$$

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4. [2350/101525 (12 pts)] Consider the function $f(x,y) = e^{1-x}\ln(1+y)$.

(a) (4 pts) Find the domain and range of $f(x,y)$.

(b) (8 pts) Find the second order Taylor polynomial for $f(x,y)$ centered at $(1,0)$ and use it to estimate the value of $\sqrt{e}\ln 2$.

SOLUTION:

(a) Domain $= \left\{(x,y)\in\mathbb{R}^2 \,\middle|\, y > -1\right\}$; Range $= \mathbb{R} = (-\infty,\infty)$

(b)

$$f_x(x,y) = -e^{1-x}\ln(1+y) \implies f_x(1,0) = 0 \qquad f_y = \frac{e^{1-x}}{1+y} \implies f_y(1,0) = 1$$

$$f_{xx} = e^{1-x}\ln(1+y) \implies f_{xx}(1,0) = 0 \qquad f_{yy} = -\frac{e^{1-x}}{(1+y)^2} \implies f_{yy}(1,0) = -1$$

$$f(1,0) = 0 \qquad f_{xy} = -\frac{e^{1-x}}{1+y} \implies f_{xy}(1,0) = -1$$

Then

$$\begin{aligned} T_2(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) \\ &\quad + \frac{1}{2!}\left[f_{xx}(1,0)(x-1)^2 + 2f_{xy}(1,0)(x-1)(y-0) + f_{yy}(1,0)(y-0)^2\right] \\ &= 0 + 0(x-1) + 1(y-0) + \frac{1}{2!}\left[0(x-1)^2 + 2(-1)(x-1)(y-0) - 1(y-0)^2\right] \\ &= y - (x-1)y - \frac{y^2}{2} = (2-x)y - \frac{1}{2}y^2 \end{aligned}$$

and

$$\sqrt{e}\ln 2 = f\left(\frac{1}{2}, 1\right) \approx T_2\left(\frac{1}{2},1\right) = \left(2 - \frac{1}{2}\right)(1) - \frac{1}{2}\left(1^2\right) = 1$$

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5. [2350/101525 (16 pts)] The following parts are unrelated.

(a) (8 pts) If $f(u,v) = u^v$ and $u = e^{st}$ and $v = st$, use the chain rule to find the rate of change of f with respect to s. Write your final answer in terms of s and t. No points awarded if the chain rule is not used.

(b) (8 pts) Let z be defined implicitly as a function of x and y by $z^3 + e^{xyz} = 2$. Find the rate of change of z with respect to x at the point $(0,3,1)$.

SOLUTION:

(a)

$$f_s = f_u u_s + f_v v_s = vu^{v-1}te^{st} + u^v(\ln u)t = st\left(e^{st}\right)^{st-1}te^{st} + \left(e^{st}\right)^{st}\left(\ln e^{st}\right)t = st^2e^{s^2t^2} + e^{s^2t^2}st^2 = 2st^2e^{s^2t^2}$$

(b)

$$3z^2\frac{\partial z}{\partial x} + e^{xyz}y\left(x\frac{\partial z}{\partial x} + z\right) = 0$$

$$\frac{\partial z}{\partial x} = \frac{-yze^{xyz}}{3z^2 + xye^{xyz}} \implies \left.\frac{\partial z}{\partial x}\right|_{(0,3,1)} = \frac{-3(1)e^{(0)(3)(1)}}{3(1)^2 + (0)(3)e^{(0)(3)(1)}} = -1$$

Alternatively, letting $F(x,y,z) = z^3 + e^{xyz}$, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{-yze^{xyz}}{3z^2 + xye^{xyz}}$$

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6. [2350/101525 (18 pts)] You need to find the maximum possible error in a first degree Taylor polynomial centered at $(1,2)$ for a certain function $f(x,y)$. You are interested in using the linear approximation in the region $\mathcal{R}$ given by $|x-1| \le 1,\ |y-2| \le 2$. Two of your friends have already discovered that the largest values of $|f_{xy}|, |f_{yy}|$ on $\mathcal{R}$ are 0.5 and 3, respectively. This means that you only have to find the maximum and minimum values of $f_{xx}(x,y) = xy - x - y$ on $\mathcal{R}$. You could simply guess some numbers for these values but that will get you zero points towards your exam grade. For notational simplicity, let's set $f_{xx}(x,y) = g(x,y) = xy - x - y$.

   (a) (4 pts) Do you know that $g(x,y)$ possesses these maximum and minimum values? Why or why not?

   (b) (10 pts) Find these maximum and minimum values.

   (c) (4 pts) Find the upper bound on the error in the first degree Taylor polynomial.

SOLUTION:

(a) Yes. $g(x,y)$ is a continuous function (polynomial) on $\mathcal{R}$, a closed and bounded region so the Extreme Value Theorem applies.

(b) Begin by finding the critical points of $g(x,y)$.

$$g_x = y - 1 = 0 \implies y = 1 \qquad g_y = x - 1 = 0 \implies x = 1 \implies \text{critical point at } (1,1) \text{ with } g(1,1) = -1$$

Now check the boundary of $\mathcal{R}$. Note $|x-1| \le 1 \implies 0 \le x \le 2$ and $|y-2| \le 2 \implies 0 \le y \le 4$.

$$x = 0,\ 0 \le y \le 4 \implies g(0,y) = -y, \text{ with minimum} = -4, \text{maximum} = 0 \quad \text{(no interior critical points)}$$

$$x = 2,\ 0 \le y \le 4 \implies g(2,y) = 2y - 2 - y = y - 2, \text{ with minimum} = -2, \text{maximum} = 2 \quad \text{(no interior critical points)}$$

$$y = 0,\ 0 \le x \le 2 \implies g(x,0) = -x, \text{ with minimum} = -2, \text{maximum} = 0 \quad \text{(no interior critical points)}$$

$$y = 4,\ 0 \le x \le 2 \implies g(x,4) = 4x - x - 4 = 3x - 4, \text{ with minimum} = -4, \text{maximum} = 2 \quad \text{(no interior critical points)}$$

So the maximum value of $g(x,y) = f_{xx}(x,y)$ is 2 and the minimum is -4. Remark: Since the function on the boundaries is linear, it has no critical points there so the Extreme Value Theorem from single variable calculus implies that the extrema must occur at the endpoints of the intervals shown.

(c) The upper bound for the error in the first degree Taylor polynomial for $f(x,y)$ centered at (a,b) is

$$|E(x,y)| \le \frac{M}{2!}\left(|x-a| + |y-b|\right)^2$$

where M is an upper bound on the absolute values of the second derivatives of f over the region $\mathcal{R}$, that is,

$$M = \max_{(x,y)\in\mathcal{R}} \left\{|f_{xx}|, |f_{xy}|, |f_{yy}|\right\} = \max\left\{0.5, 3, 4\right\} = 4$$

Thus, with $|x-1| \le 1$ and $|y-2| \le 2$

$$|E(x,y)| \le \frac{4}{2!}\left(1+2\right)^2 = 2(3)^2 = 18$$

Alternatively, we could also use the following error bound, utilizing the values from above for the absolute values of the second derivatives:

$$|E(x,y)| \leq \frac{1}{2!}\left[|f_{xx}(x,y)||x-1|^2 + 2|f_{xy}(x,y)||x-1||y-2| + |f_{yy}||y-2|^2\right] \quad \text{for } (x,y) \in \mathcal{R}$$

$$\leq \frac{1}{2}\left(4(1)^2 + 2(0.5)(1)(2) + 3(2)^2\right) = 9$$

■