1. [2360/101525 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

   (a) If $\lambda = 0$ is an eigenvalue of the square matrix $\mathbf{A}$, then the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ must be inconsistent.

   (b) The column space of an $n \times n$ nonsingular matrix is $\mathbb{R}^n$.

   (c) Given any $m \times n$ matrix $\mathbf{A}$, you can always compute $\operatorname{Tr}\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)$ and $\operatorname{Tr}\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\right)$.

   (d) Every set of n vectors in an n-dimensional vector space forms a basis.

   (e) If $\mathbf{A}$ is an $m \times n$ matrix and $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are solutions of $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$, then $\vec{\mathbf{u}} - \vec{\mathbf{v}}$ is a solution of the associated homogeneous system.

SOLUTION:

(a) **FALSE** The system could have infinitely many solutions.

(b) **TRUE** $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a solution for every $\vec{\mathbf{b}} \in \mathbb{R}^n \implies \operatorname{Col}\mathbf{A} = \mathbb{R}^n$

(c) **TRUE** Both $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ are square so the trace can be computed.

(d) **FALSE** The n vectors must be linearly independent in order to form a basis.

(e) **TRUE** $\mathbf{A}(\vec{\mathbf{u}} - \vec{\mathbf{v}}) = \mathbf{A}\vec{\mathbf{u}} - \mathbf{A}\vec{\mathbf{v}} = \vec{\mathbf{b}} - \vec{\mathbf{b}} = \vec{\mathbf{0}}$

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2. [2360/101525 (18 pts)] Consider the following matrices: $\mathbf{F} = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -2 & 3 \end{bmatrix}$, $\mathbf{G} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\mathbf{H} = \begin{bmatrix} 2 & -1 \end{bmatrix}$. Compute the following if possible, making sure to fully simplify the result. If not possible to compute, write NP.

   (a) $|\mathbf{G}|$ (b) $\mathbf{H}\mathbf{F}\mathbf{F}^{\mathrm{T}}$ (c) $\operatorname{Tr}\mathbf{H}$ (d) $\mathbf{G}^{\mathrm{T}}\mathbf{G}$ (e) $5\mathbf{F} - 8\mathbf{G}$ (f) $\left|\mathbf{H}^2\right|$

SOLUTION:

(a) $\cos^2\theta + \sin^2\theta = 1$

(b)

$$\begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 0 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -4 & 6 & -3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 16 & -29 \end{bmatrix}$$

(c) NP

(d)

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

(e) NP

(f) NP

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3. [2360/101525 (15 pts)] Use Gauss-Jordan row reduction to find the RREF and solution of the following system.

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 7 \\ x_1 + x_2 + x_3 &= 6 \\ 2x_1 + 5x_3 &= 19 \\ 3x_1 - x_2 + 2x_3 &= 11 \end{aligned}$$

SOLUTION:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 1 & 1 & 6 \\ 2 & 0 & 5 & 19 \\ 3 & -1 & 2 & 11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & -1 & 0 & -1 \\ 0 & -4 & 3 & 5 \\ 0 & -7 & -1 & -10 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & -1 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{x}} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

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4. [2360/101525 (12 pts)] Find all of the eigenvalues and eigenvectors of $\mathbf{B} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$. State the algebraic and geometric multiplicity of each eigenvalue.

SOLUTION:

$$\begin{vmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = (2-\lambda)(-1)^{2+2} \begin{vmatrix} -\lambda & 2 \\ 2 & -\lambda \end{vmatrix} = (2-\lambda)\left(\lambda^2 - 4\right) = -(\lambda-2)^2(\lambda+2) = 0 \implies \lambda = 2, -2$$

$$\lambda = -2: \ (\mathbf{B} + 2\mathbf{I})\vec{\mathbf{v}} = \mathbf{0} \implies \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 4 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

algebraic multiplicity $= 1$
geometric multiplicity $= 1$

$$\lambda = 2: \ (\mathbf{B} - 2\mathbf{I})\vec{\mathbf{v}} = \mathbf{0} \implies \left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

algebraic multiplicity $= 2$
geometric multiplicity $= 2$

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5. [2360/101525 (15 pts)] This problem deals with the vector space $\mathbb{P}_2$.

   (a) (7 pts) Does $\mathbb{P}_2 = \text{span}\left\{3t^2 - 4, 2t, t^2 - 1\right\}$? Draw your conclusion, if possible, by computing the Wronksian.

   (b) (8 pts) Without using the Wronskian, determine if the set $\left\{2 + 2t^2, -1 + 4t + 3t^2, 3 - 5t - 2t^2\right\}$ forms a basis for $\mathbb{P}_2$.

SOLUTION:

(a) Yes. The set consists of three linearly independent vectors in the 3-dimensional space. Justification:

$$W(t) = \begin{vmatrix} 3t^2 - 4 & 2t & t^2 - 1 \\ 6t & 2 & 2t \\ 6 & 0 & 2 \end{vmatrix} = 6(-1)^{3+1} \begin{vmatrix} 2t & t^2 - 1 \\ 2 & 2t \end{vmatrix} + 2(-1)^{3+3} \begin{vmatrix} 3t^2 - 4 & 2t \\ 6t & 2 \end{vmatrix}$$

$$= 6\left(4t^2 - 2t^2 + 2\right) + 2\left(6t^2 - 8 - 12t^2\right) = 12t^2 + 12 - 12t^2 - 16 = -4 \neq 0$$

(b) No. Although there are three vectors in the set, they are linearly dependent and thus cannot form a basis. Justification:

$$c_1\left(2 + 2t^2\right) + c_2\left(-1 + 4t + 3t^2\right) + c_3\left(3 - 5t - 2t^2\right) = 0 + 0t + 0t^2$$

$$\text{equating coefficients} \implies \begin{aligned} t^0 &: \ 2c_1 - 1c_2 + 3c_3 = 0 \\ t^1 &: \ 0c_1 + 4c_2 - 5c_3 = 0 \\ t^2 &: \ 2c_1 + 3c_2 - 2c_3 = 0 \end{aligned} \implies \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ 2 & 3 & -2 \end{vmatrix} = 2(-1)^{1+1} \begin{vmatrix} 4 & -5 \\ 3 & -2 \end{vmatrix} + 2(-1)^{3+1} \begin{vmatrix} -1 & 3 \\ 4 & -5 \end{vmatrix} = 2(-8+15) + 2(5-12) = 2(7) + 2(-7) = 0$$

This shows that the system possesses nontrivial solutions and thus the functions are linearly dependent.

Alternatively, the RREF of the coefficient matrix is $\begin{bmatrix} 1 & 0 & \frac{7}{8} \\ 0 & 1 & -\frac{5}{4} \\ 0 & 0 & 0 \end{bmatrix}$ from which the same conclusion follows.

6. [2360/101525 (12 pts)] The following parts are not related.

(a) (6 pts) Let $\mathbf{C}$ and $\mathbf{D}$ be invertible matrices. Solve for $\vec{\mathbf{x}}$ if $\mathbf{C}(\mathbf{DC})^{-1}\vec{\mathbf{x}} = \vec{\mathbf{y}}$. Simplify your answer completely.

(b) (6 pts) For which value(s) of k, if any, will the system $\mathbf{A}^{\mathrm{T}}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ have a unique solution if $\mathbf{A} = \begin{bmatrix} 1 & 0 & k \\ 0 & k & 1 \\ k & 0 & 4 \end{bmatrix}$.

SOLUTION:

(a)

$$\vec{\mathbf{x}} = \left[\mathbf{C}\left(\mathbf{DC}\right)^{-1}\right]^{-1}\vec{\mathbf{y}} = \left[(\mathbf{DC})^{-1}\right]^{-1}\mathbf{C}^{-1}\vec{\mathbf{y}} = \mathbf{DCC}^{-1}\vec{\mathbf{y}} = \mathbf{DI}\vec{\mathbf{y}} = \mathbf{D}\vec{\mathbf{y}}$$

(b) To have a unique solution, $\mathbf{A}^{\mathrm{T}}$ must be invertible. This is equivalent to saying that $|\mathbf{A}^{\mathrm{T}}| \neq 0$ or $|\mathbf{A}| \neq 0$. Now

$$|\mathbf{A}| = k(-1)^{2+2}\begin{vmatrix} 1 & k \\ k & 4 \end{vmatrix} = k(4-k^2) = k(2-k)(2+k)$$

$\mathbf{A}^{\mathrm{T}}$ will be invertible and thus the system will have a unique solution if and only if $k \neq -2, 0, 2$.

7. [2360/101525 (18 pts)] In each of the following problems, decide whether the given subset $\mathbb{W}$ of the vector space $\mathbb{V}$ is or is not a subspace of $\mathbb{V}$. Justify your answers. If not a subspace, identify at least one requirement that is not satisfied. Assume the standard definitions of vector addition and scalar multiplication in each vector space.

(a) (6 pts) $\mathbb{V} = \mathbb{R}^2,\ \mathbb{W} = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, y = px^2, p \in \mathbb{R}\right\}$.

(b) (6 pts) $\mathbb{V} = \mathbb{M}_{22},\ \mathbb{W}$ is the set of 2×2 matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ with $a+b=c$ and $a,b,c \in \mathbb{R}$.

(c) (6 pts) $\mathbb{V} = \mathcal{C}(-\infty,\infty),\ \mathbb{W}$ is the set of all negative functions in $\mathcal{C}(-\infty,\infty)$.

SOLUTION:

(a) Not a subspace. Some possible justifications, choosing $p=1$:

- Not closed under vector addition. $(1,1),(2,4) \in \mathbb{W}$ but $(1,1)+(2,4)=(3,5) \notin \mathbb{W}$
- Not closed under scalar multiplication. $(1,1) \in \mathbb{W}$ but $2(1,1)=(2,2) \notin \mathbb{W}$
- Not closed under linear combinations. Let $\vec{\mathbf{u}} = \left(a,a^2\right), \vec{\mathbf{v}} = \left(b,b^2\right) \in \mathbb{W}$ and $c,d \in \mathbb{R}$. Then

$$c\vec{\mathbf{u}} + d\vec{\mathbf{v}} = c\left(a,a^2\right) + d\left(b,b^2\right) = \left(ca,ca^2\right) + \left(db,db^2\right) = \left(ca+db, ca^2+db^2\right) \notin \mathbb{W} \text{ since } ca^2+db^2 \neq (ca+db)^2$$

(b) Subspace. Let $\vec{\mathbf{u}} = \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix}$ be in $\mathbb{W}$. This means that $a_1+b_1=c_1$ and $a_2+b_2=c_2$. Let $r,s \in \mathbb{R}$. Then

$$r\vec{\mathbf{u}} + s\vec{\mathbf{v}} = r\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} + s\begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} ra_1 & rb_1 \\ 0 & rc_1 \end{bmatrix} + \begin{bmatrix} sa_2 & sb_2 \\ 0 & sc_2 \end{bmatrix} = \begin{bmatrix} ra_1+sa_2 & rb_1+sb_2 \\ 0 & rc_1+sc_2 \end{bmatrix} \text{ with}$$

$$(ra_1+sa_2)+(rb_1+sb_2) = r(a_1+b_1)+s(a_2+b_2) = rc_1+sc_2 \implies r\vec{\mathbf{u}}+s\vec{\mathbf{v}} \in \mathbb{W}$$

(c) Not a subspace. Some possible justifications:

- The zero vector, that is the function that is zero for all real numbers, is not in $\mathbb{W}$.
- $f(t) = -1-t^2 \in \mathbb{W}$ but $-1f(t) = 1+t^2 \notin \mathbb{W}$, giving an example of nonclosure with respect to scalar multiplication.
- More generally $\mathbb{W}$ is not closed under scalar multiplication. If $f(t) \in \mathbb{W}$, then $f(t) < 0$ for all t. But $-1f(t) > 0$ and thus not in $\mathbb{W}$.

Note that $\mathbb{W}$ is closed under vector addition.