APPM 3570/STAT 3100 — Exam 1 — Fall 2025

On the front of your bluebook, write (1) **your name**, (2) **Exam 1**, (3) **APPM 3570/STAT 3100**. Correct answers with no supporting work may receive little or no credit. Books, notes and electronic devices of any kind are not allowed. Your exam should be uploaded to Gradescope in a PDF format (Recommended: **Genius Scan**, **Scannable** or **CamScanner** for iOS/Android). **Show all work, justify your answers.** **Do all problems.** Students are required to re-write the **honor code statement** in the box below on the **first page** of their exam submission and **sign and date it**:

On my honor, as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work.
Signature:____________________________ Date:_________

1. [Exam01] (40pts) There are 4 unrelated parts to this question. Justify your answers.

   (a) (10pts) How many ways are there to place 12 marbles of the same size in five distinct urns: (i)(5pts) if the marbles are all black? (ii)(5pts) if each marble is a different color?

   (b) (10pts) If $P(A) = 0.6$, $P(B) = 0.3$, $P(A \cap B^c) = 0.4$ and $B \subseteq C$, calculate $P(A \cup B^c \cup C^c)$.

   (c) (10pts) (i)(5pts) How many different permutations of the letters in the word W A K A T A K A K A G E are there? (ii)(5pts) If you randomly select one of these permutations, what is the probability you will select a permutation that *exactly* contains the string W A K A in that order, in any part of the permutation?

   (d) (10pts) An ice cream shop has a total of seven flavors of ice cream (including chocolate and vanilla). (i)(5pts) Consider the experiment of selecting two flavors of ice cream, how many total possible outcomes are there? (ii)(5pts) Ralphie plans on getting a bowl with two *different* scoops of ice cream. What is the probability that one of the scoops she chooses will be vanilla or chocolate?

Solution:

(a)(i)(5pts) If the marbles are all black, then they are not distinct and each solution of $x_1 + x_2 + \cdots + x_5 = 12$ corresponds to one way to distribute the marbles in five different urns and vice-versa. Thus, the number of ways to distribute the marbles is $\frac{12+(5-1)}{12!4!} = \binom{16}{12} = 1{,}820$.

(a)(ii)(5pts) In this case, if we consider the point of view of the distinct marbles, we have five options for the first marble *and* five options for the next marble, etc. So we have $5^{12} = 244{,}140{,}625$ ways to distribute 12 different colored marbles into five distinct urns.

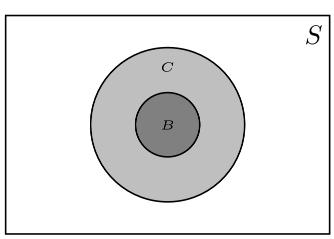
Method 2: Note that $\binom{12}{n_1,n_2,n_3,n_4,n_5}$ where each solution of $n_1 + n_2 + \cdots + n_5 = 12$ represents ways to distribute the 12 marbles into the five urns. In total, by the Multinomial Theorem, we have

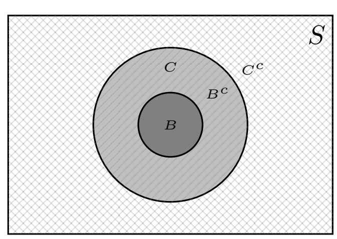
$$\sum_{n_1+n_2+\cdots+n_5=12} \binom{12}{n_1,n_2,n_3,n_4,n_5} = \sum_{n_1+n_2+\cdots+n_5=12} \binom{12}{n_1,n_2,n_3,n_4,n_5} 1^{n_1}1^{n_2}\cdots 1^{n_5} \overset{*}{=} (1+1+1+1+1)^{12} = 5^{12}.$$

$(*)$ We claim that $(x_1, x_2, x_3, x_4, x_5) = (1, 1, 1, 1, 1)$ is the only combination of values that satisfies the equation above for *all* choices of $(n_1, n_2, n_3, n_4, n_5)$ such that $n_1 + n_2 + \cdots + n_5 = 12$.

(b)(10pts) If $B \subseteq C$ then $C^c \subseteq B^c$ and so $A \cup B^c \cup C^c = A \cup B^c$. Thus, by the Inclusion-Exclusion Principle, we have

$$P(A \cup B^c \cup C^c) = P(A \cup B^c) = P(A) + P(B^c) - P(A \cap B^c) = 0.6 + (1 - 0.3) - 0.4 = 0.9.$$





(c)(i)(5pts) *Hakkeyoi!* The word WAKATAKAKAGE has 12 letters, with five A's, and three K's, thus, adjusting for overcount, we have

$$\frac{12!}{5!3!} = \binom{12}{5,3,1,1,1,1} = 665{,}280 \text{ permutations of the letters in the word WAKATAKAKAGE.}$$

(c)(ii)(5pts) Consider the "superletter" WAKA and the remaining eight letters T A K A K A G E of which there are $\frac{8!}{3!2!}$ permutations and, for each one of these permutations, there are nine possible positions for the string WAKA, thus, we have $9 \cdot \frac{8!}{3!2!} = \frac{9!}{3!2!}$ permutations that exactly contain the string WAKA. The probability of selecting a permutation of the letters W A K A T A K A K A G E that exactly contains the string WAKA is

$$P(\text{WAKA-string}) = \frac{9!/3!2!}{12!/5!3!} = \frac{9!5!}{12!2!} = \frac{5 \cdot 4 \cdot 3}{12 \cdot 11 \cdot 10} = \frac{1}{22} = 0.04545\ldots.$$

(d)(5pts)(i) Consider the experiment of selecting two flavors of ice cream. We assume the most general case and allow for two scoops of the same flavor (since no restriction is mentioned). The order in which ice cream is scooped does not matter, that is, order does not matter. Since order does not matter, the size of the sample space S is $|S| = 7 + \frac{7 \cdot 6}{2} = 7 + 21 = 28$.

(d)(5pts)(ii) Ralphie plans on getting a bowl with two *different* scoops of ice cream. Let C be the event that one of the scoops selected is chocolate the other is not and, similarly, let V be the event that exactly one of the two scoops selected is vanilla. The event $C \cup V$ is the event that Ralphie chooses two different scoops of ice cream where one of them is chocolate or vanilla and, by the Inclusion-Exclusion Principle, we have

$$P(C \cup V) = P(C) + P(V) - P(C \cap V) = \frac{1 \cdot \binom{6}{1}}{28} + \frac{1 \cdot \binom{6}{1}}{28} - \frac{1}{28} = \frac{11}{28} \approx 0.3929.$$

2. [Exam01] (32pts) Customers who purchase vehicles at a certain dealership can order an engine in any of three sizes: *small*, *medium*, *large*. Of all cars sold at this dealership, 50% have the small engine and 30% have a medium-sized engine. Of cars with a medium sized engine, 20% fail emissions test within two years of purchase, while only 10% of cars fail emissions test within two years of purchase if they have the small engine. The percentage of vehicles that have the large engine *and* pass the emissions test within two years of purchase is 12%. *(Answer the questions below, justify your answers.)*

   (a) (8pts) Given that a vehicle from the dealership has a large engine, what is the probability it will pass the emissions tests within two years of purchase?

   (b) (8pts) What is the probability that a randomly chosen car from the dealership will fail an emissions test within two years of purchase?

   (c) (8pts) Of the next 100 cars sold at the dealership, what is the probability that 60 of them will fail an emissions test within two years of purchase? *State any assumptions you have made.*

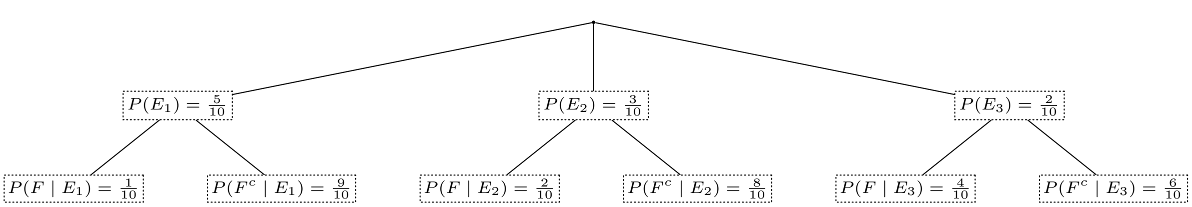
   (d) (8pts) A vehicle that *passed* the emissions test is chosen at random, what is the probability that it has a small engine?

Solution:

Let E_1, E_2 and E_3 denote the event that a car has a small, medium and large engine respectively. Let F be the event that a car fails an emissions test within two years.

(a)(8pts) Of all cars sold, 50% have the smallest engine and 30% have a medium-sized engine so 20% of the vehicles sold have large engines, that is, $P(E_3) = \frac{20}{100}$. Note $E_3 \cap F^c$ is the event that a vehicle has a large engine *and* passes the emissions test so $P(E_3 \cap F^c) = \frac{12}{100}$, thus, the conditional probability that a vehicle passes the emissions test within two years of purchase given that it is a large vehicle is $P(F^c|E_3) = \frac{P(E_3 \cap F^c)}{P(E_3)} = \frac{12/100}{20/100} = \frac{6}{10}$.

Thus, we have the following probabilities:



(b)(8pts) Note that $F = F \cap S = F \cap (E_1 \cup E_2 \cup E_3) = FE_1 \cup FE_2 \cup FE_3$, thus, by the Law of Total Probability, we have

$$\begin{aligned} P(F) &= P(FE_1 \cup FE_2 \cup FE_3) \\ &= P(F \cap E_1) + P(F \cap E_2) + P(F \cap E_3) \\ &= P(E_1)P(F|E_1) + P(E_2)P(F|E_2) + P(E_3)P(F|E_3) \\ &= \frac{5}{10} \cdot \frac{1}{10} + \frac{3}{10} \cdot \frac{2}{10} + \frac{2}{10} \cdot \frac{4}{10} = \frac{5+6+8}{100} = \frac{19}{100}. \end{aligned}$$

(c)(8pts) Let $p = P(F)$, and let X be the number of cars out of 100 that fail the emissions test within the next two years then the number of ways exactly 60 vehicles out of 100 can fail is $\binom{100}{60}$ and, assuming each vehicle passes or fails the emissions test **independently** of the other, the probability of exactly 60 vehicles failing is $p^{60}(1-p)^{40}$ by independence. Thus, the probability that 60 out of 100 cars from dealership will fail an emissions test within two years of purchase is

$$P(X = 60) = \binom{100}{60} p^{60}(1-p)^{40}.$$

(d)(8pts) Using Bayes Rule (reverse conditioning) we have

$$P(E_1|F^c) = \frac{P(E_1 \cap F^c)}{P(F^c)} = \frac{P(E_1)P(F^c|E_1)}{1 - P(F)} = \frac{\frac{5}{10} \cdot \frac{9}{10}}{1 - \frac{19}{100}} = \frac{45}{81} = 0.5555\ldots.$$

3. [Exam01] (28pts) An experiment consists of flipping a coin until the first head appears or until a total of 5 flips is made. Assume the probability of getting a head on each flip is $1-q$ and the probability of a tail on each flip is $q \in [0,1]$.

   (a) (7pts) What is the sample space for this experiment? **Give your answer in set notation.**

   (b) (7pts) Let Y be the random variable that counts the number of flips. Find the *probability mass function* (pmf) of Y. (Your pmf should be defined for all real numbers.) *State any assumptions you are making.*

   (c) (7pts) Verify that your answer from part (b) is indeed a probability mass function. Show all work.

   (d) (7pts) For what value of $q \in [0,1]$ will the likelihood of event $\{Y = 4\}$ occurring be maximized? Justify your answer.

Solution:

(a)(7pts) If H denotes the event of getting a head and T denotes the event of flipping a tail, then we can write the sample space as

$$\mathcal{S} = \{H, TH, TTH, TTTH, TTTTH, TTTTT\}.$$

(b)(7pts) If Y is the number of flips, then $Y \in \{1,2,3,4,5\}$ and, assuming each flip is **independent**, we have

$$\begin{aligned} &p(1) = P(Y=1) = P(\{H\}) = 1-q, \\ &p(2) = P(Y=2) = P(\{TH\}) = q(1-q), \\ &p(3) = P(Y=3) = P(\{TTH\}) = q^2(1-q), \\ &p(4) = P(Y=4) = P(\{TTTH\}) = q^3(1-q), \\ &p(5) = P(Y=5) = P(\{TTTTH\} \cup \{TTTTT\}) = P(\{TTTTH\}) + P(\{TTTTTT\}) = q^4(1-q) + q^5 = q^4 \end{aligned}$$

and $p(k) = 0$ for all other real values of k.

(c)(7pts) Observe that

$$\sum_{i \in \mathbb{R}} p(i) = \sum_{i=1}^{5} p(i) = (1-q) + q(1-q) + q^2(1-q) + q^3(1-q) + q^4 = (1-q) + (q-q^2) + (q^2-q^3) + (q^3-q^4) + q^4 = 1.$$

Note that $0 \le q \le 1$ so $0 \le q^2 \le q$ which implies $q^2 \in [0,1]$ and in general, $q^k \in [0,1]$ for $k = 0, 1, \ldots$. Also, if $q \in [0,1]$ then $1-q \in [0,1]$ and, finally, we can show $q^k(1-q) \in [0,1]$, that is, $p(k) \in [0,1]$ for all $k \in \mathbb{R}$. Thus, the function $p(k)$ as define in part (b) is verified to be a pmf.

(d)(7pts) The "likelihood of event $\{Y=4\}$ occurring" is the probability $P(Y=4)$. Note that $P(Y=4) = f(q)$ where $f(q) = q^3(1-q) = q^3 - q^4$. We can maximize $f(q) = q^3 - q^4$, where $q \in [0,1]$, using Calculus. Note that $f'(q) = 3q^2 - 4q^3 = q^2(3-4q)$ and $f'(q) = 0$ implies $q = 0, \frac{3}{4}$. Since $f(0) = f(1) = 0$ and $f(\frac{3}{4}) > 0$, the likelihood of event $\{Y=4\}$ occurring is maximized when $q = \frac{3}{4}$.