Write your name and your professor's name or your section number below. You are *not* allowed to use textbooks, notes, or a calculator. To receive full credit on a problem you must show **sufficient justification for your conclusion** unless explicitly stated otherwise.

Name:

Instructor and Section:

1. (24 pts) If the statement is **always true**, write “TRUE”; if it is possible for the statement to be false then mark “FALSE.” You must give a **justification** for your answer. That is, if the answer is true, provide a brief proof. If the answer is false, provide a counterexample.

   (a) The vectors $\vec{v}_1 = (1, 0, 1)^T$, $\vec{v}_2 = (0, 1, 1)^T$, and $\vec{v}_3 = (0, 0, 1)^T$ form a basis for $\mathbb{R}^3$.

   (b) If A and B are nonsingular $n \times n$ matrices, then $(A + B)^{-1} = A^{-1} + B^{-1}$

   (c) If A and B are symmetric $n \times n$ matrices then AB is symmetric.

   (d) The set of vectors $(x, y, z)^T$ with $x, y, z \in \mathbb{R}$ such that $x + y + z = 1$ is a vector subspace.

Solution

(a) TRUE. Let $A = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ with columns $\vec{v}_i$. It is not hard to see that $rank(A) = 3$, and so these vectors span $\mathbb{R}^3$. Moreover 3 is the minimal number needed to span this 3D space, so they form a basis. You should show the rank by row reduction.

(b) FALSE We need a counterexample. There are many. One is let $A = I$ and $B = -I$. they are each invertible, but $A + B = 0$ is not even invertible. So the equality cannot possibly be true. You can also consider 1×1 matrices to get a counterexample.

(c) FALSE We know that $A^T = A$ and $B^T = B$, and that generaly $(AB)^T = B^T A^T$. So for this case, $(AB)^T = BA$. It is not true generally that $AB = BA$, i.e., the matrices commute. A simple example is $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. These are indivdually symmetric, but $(AB)_{12} = 1$, but $(AB)_{21} = 2$. So AB is not symmetric.

(d) FALSE Every vector space **must** contain the zero vector. However, for $\vec{v} = (0, 0, 0)^T$, we would have $x + y + z = 0$, which is not in this set. So it cannot be a vector subspace. One can also see that a linear combination of points is not in the set, e.g., $(1, 0, 0)$ and $(0, 1, 0)$ are both in the set, but their sum $(1, 1, 0)$ is no longer in the set.

2. (21 pts) Let A be the matrix

$$A = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & c & 1 \end{pmatrix}$$

for c an arbitrary real parameter.

(a) (5 pts) Reduce A to a matrix U in Row-Echelon form.

(b) (5 pts) What is rank(A)? How does it depend on c?

(c) (5 pts) Solve the homogeneous problem $A\vec{x} = \vec{0}$. Does $A\vec{x} = \vec{0}$ always have a solution? Explain why or why not.

(d) (6 pts) Solve the inhomogeneous problem $A\vec{x} = (1, 1, 1)^T$. Does this have a solution for all values of c? Explain why or why not.

Solution

(a) Knowing that (d) is coming, let's consider the augmented matrix $(A|b)$ with $\vec{b} = (1, 1, 1)^T$. The operations to reduce this are $-2R_1 + R_2 \to R_2$, then $-3R_1 + R_3 \to R_3$, and then finally $-2R_2 + R_3 \to R_3$

$$A|b = \begin{pmatrix} 1 & 0 & 3 & 1 & |1 \\ 2 & 1 & 1 & 1 & |1 \\ 3 & 2 & c & 1 & |1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 1 & 1 \\ 0 & 1 & -5 & -1 & -1 \\ 0 & 2 & c-9 & -2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 1 & 1 \\ 0 & 1 & -5 & -1 & -1 \\ 0 & 0 & c+1 & 0 & 0 \end{pmatrix}$$

So for this part we get

$$U = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -5 & -1 \\ 0 & 0 & c+1 & 0 \end{pmatrix}$$

(b) When $c \neq -1$ there are three pivots in U, so $rank(A) = 3$ However, when $c = -1$, there are only two pivots, so $rank(A) = 2$.

(c) For this we solve the homogeneous problem $A\vec{z} = \vec{0}$. When $c \neq -1$, there is only one free variable, $z_4 = \alpha$, say, giving the general solution $\vec{z}_1 = \alpha(-1, 1, 0, 1)^T$. However, when $c = -1$, there are two free variables, $z_3 = \beta$ and $z_4 = \alpha$. Solving gives $\vec{z}_2 = \alpha(-1, 1, 0, 1)^T + \beta(-3, 5, 1, 0)^T$. In both cases there are solutions. Recall that the homogeneous problem $A\vec{z} = \vec{0}$ ALWAYS has a solution, but if $rank(A) = n$, the only solution would be zero. Here $rank(A) < n = 4$, so $\ker(A)$ is one or two dimensional, respectively.

(d) For the inhomogeneous problem we use the augmented matrix above. This gives

$$(c \neq -1): \quad \vec{x} = (1, -1, 0, 0)^T + \vec{z}_1$$
$$(c = -1): \quad \vec{x} = ((1, -1, 0, 0)^T + \vec{z}_2$$

where $\vec{z}_1$ and $\vec{z}_2$ are the homogenous solutions found above. Note again there are always solutions because the system is consistent, even when $c = -1$, because then the last row would be zero.

3. (15 pts)

   (a) Construct a basis for the vector space of real skew-symmetric 3×3 matrices, $A^T = -A$. What is the dimension of this space?

   (b) Do the same (i.e., construct a basis) for the vector space of real symmetric 3×3 matrices, $A^T = A$. What is its dimension?

   (c) Any real 3×3 matrix can be written as a sum of symmetric and skew-symmetric matrices,

$$A = \tfrac{1}{2}\left(A + A^T\right) + \tfrac{1}{2}\left(A - A^T\right).$$

   What is the dimension of the vector space of all real 3×3 matrices?

Solution:

(a) The vector space of real skew-symmetric 3×3 matrices has dimension $d = 3$ and a basis is given by the following three matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

(b) The vector space of real symmetric 3×3 matrices has dimension $d = 6$ and a basis is given by six matrices,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(c) The dimension of the vector space of all real 3×3 matrices is $d = 9$. A possible basis for this space is given by the set of all the matrices listed above (which are linearly independent).

4. (20 pts) For the matrix

$$A = \begin{pmatrix} 0 & 2 & -2 & 2 \\ -2 & 1 & -1 & -1 \\ -1 & 1 & -1 & 0 \end{pmatrix}$$

find bases for the

(a) (7 pts) image of A and

(b) (7 pts) kernel of A.

(c) (6 pts) Is $(6, -1, 1)^T$ in the image of A?

Solution: To find these vector subspaces, we must first find the REF of A:

$$\begin{pmatrix} 0 & 2 & -2 & 2 \\ -2 & 1 & -1 & -1 \\ -1 & 1 & -1 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} -1 & 1 & -1 & 0 \\ -2 & 1 & -1 & -1 \\ 0 & 2 & -2 & 2 \end{pmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 2 & -2 & 2 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \begin{pmatrix} -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

And we can continue reducing to RREF if we'd like:

$$\xrightarrow{R_1 \to R_1 + R_2} \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) We take the pivot columns (the first two columns) of A for the basis of img$A$.:

$$\left\{ \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$$

(b) For our basis of the kernel, the matrix is rank 2 and has four columns. We therefore need to find two linearly independent vectors that solve $A\vec{z} = \vec{0}$ First choose $z_3 = 1$ and $z_2 = 0$ and solve for z_1 and z_2:

$$\begin{aligned} 0 - z_2 + 1 + 0 &= 0 && \text{(Row 2)} \\ z_2 &= 1 \\ -z_1 + 0 + 0 + 0 &= 0 && \text{(Row 1)} \\ z_1 &= 0 \end{aligned}$$

So our first basis vector is

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Now we set $z_3 = 0$ and $z_4 = 1$ to guarantee we have linearly dependent vectors. Solving for the other two elements gives

$$\begin{aligned} 0 - z_2 + 0 - 1 &= 0 && \text{(Row 2)} \\ z_2 &= -1 \\ -z_1 + 0 + 0 - 1 &= 0 && \text{(Row 1)} \\ z_1 &= -1 \end{aligned}$$

So our second basis vector is

$$\begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

and our basis for $\ker A$ is

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(c) The easiest way to determine if $(6, -1, 1)^T$ is in the image of A is to see if there is a linear combination of the first two columns of A that can construct this vector. In other words, solve the following system for $\vec{c}$

$$\begin{bmatrix} 0 & 2 \\ -2 & 1 \\ -1 & 1 \end{bmatrix} \vec{c} = \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}$$

The solution to this system is $\vec{c} = (2, 3)^T$, and thus $\boxed{\text{yes}}$, the vector $(6, -1, 1)^T$ is in the image of A.

5. (20 pts) Let

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 0 \end{pmatrix}.$$

(a) (7 pts) Find the permutation matrix P such that PA is symmetric.

(b) (7 pts) Find matrices L and D such that $PA = LDL^T$.

(c) (6 pts) What is the determinant of A?

Solution:

(a) Of the six possibilities, we find the only arrangement of rows that produces a symmetric matrix is exchanging R_1 and R_3, giving

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

So we have

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(b) We first factor PA into LU:

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

We then factor U into DV by putting the pivots onto the diagonal of D and dividing each row of U by it's pivot:

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

So we have

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$
$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Since PA is a symmetric matrix, the LDV factorization equals LDL^T.

(c) Since we did one row interchange, the determinant of A is minus the determinant of D, and the determinant if a diagonal matrix is simply the product of its entries:

$$\det A = (-1)(-3) = 3$$