1. A physicist observes that the position, s measured in feet, of a certain object depends on time, t measured in seconds. The physicist establishes that the relationship is linear with equation: $s(t) = \frac{3}{2}t + 4$. (9 pts)

   (a) What are the units of the slope of the linear equation?

   Solution: $\boxed{\text{ft}/s}$ since slope $= \frac{s(t_2)-s(t_1)}{t_2-t_1}$ $\left(\frac{\text{rise}}{\text{run}}\right)$.

   (b) How far does the object move every 2 seconds?

   Solution: $\boxed{3\text{ ft}}$ since every 2 seconds the object movies $\frac{3}{2} \cdot 2 = 3$ft.

   (c) What was the position of the object at time $t = 0$ seconds?

   Solution: $\boxed{4\text{ ft}}$ which can be found by plugging $t = 0$ into the $s(t)$ equation. At $t = 0$ seconds, $s(0) = \frac{3}{2} \cdot 0 + 4 = 4$.

2. Solve the following equations: (10 pts)

   (a) $\sqrt{2x} + 3 = x - 1$

   Solution:

$$\sqrt{2x} + 3 = x - 1 \tag{1}$$
$$\sqrt{2x} + 3 - 3 = x - 1 - 3 \tag{2}$$
$$\sqrt{2x} = x - 4 \tag{3}$$

Square both sides to get:

$$(\sqrt{2x})^2 = (x - 4)^2 \tag{4}$$
$$2x = (x - 4) \cdot (x - 4) \tag{5}$$
$$2x = x^2 - 4x - 4x + 4^2 \tag{6}$$
$$2x = x^2 - 8x + 16 \tag{7}$$
$$2x - 2x = x^2 - 8x + 16 - 2x \tag{8}$$
$$0 = x^2 - 10x + 16 \tag{9}$$
$$0 = (x - 8)(x - 2) \tag{10}$$

Using zero product property,

$$x = 8, \quad x = 2. \tag{11}$$

Squaring the equation may introduce false solution(s). We need to verify our solutions.
At $x = 8$, the left side of the equation is

$$\sqrt{2 \cdot 8} + 3 = \sqrt{16} + 3 = \sqrt{4^2} + 3 = 4 + 3 = 7 \tag{12}$$

The right side is

$$8 - 1 = 7. \tag{13}$$

At $x = 2$, the left side of the equation is

$$\sqrt{2 \cdot 2} + 3 = \sqrt{2^2} + 3 = 2 + 3 = 5 \tag{14}$$

The right side is

$$2 - 1 = 1. \tag{15}$$

$\boxed{x = 8}$ is the only solution.

(b) $\dfrac{2x}{x^2-4} = \dfrac{1}{2\left(x^2-4\right)} - \dfrac{1}{x+2}$

Solution: First we factor the denominators:

$$\frac{2x}{x^2-4} = \frac{1}{2\left(x^2-4\right)} - \frac{1}{x+2} \tag{16}$$
$$\frac{2x}{(x-2)(x+2)} = \frac{1}{2(x-2)(x+2)} - \frac{1}{x+2} \tag{17}$$

Now we can multiply both sides of the equation by the least common denominator $2(x-2)(x+2)$ to get

$$2(x-2)(x+2)\cdot\frac{2x}{(x-2)(x+2)} = 2(x-2)(x+2)\left(\frac{1}{2(x-2)(x+2)} - \frac{1}{x+2}\right) \tag{18}$$
$$4x = 2(x-2)(x+2)\cdot\frac{1}{2(x-2)(x+2)} - 2(x-2)(x+2)\frac{1}{(x+2)} \tag{19}$$
$$4x = 1 - 2(x-2) \tag{20}$$
$$4x = 1 - 2x + 4 \tag{21}$$
$$4x + 2x = 5 - 2x + 2x \tag{22}$$
$$6x = 5 \tag{23}$$
$$x = \frac{5}{6} \tag{24}$$

As $x = \dfrac{5}{6}$ does not result in division by zero in the original equation then, $\boxed{x = \dfrac{5}{6}}$ is the solution.

3. Solve the following inequalities. Justify your answers by using a number line or sign chart when appropriate. Answers without full justification will not receive full credit. Express all answers in interval notation. (9 pts)

   (a) $x^4 + 14x^3 + 33x^2 < 0$

   Solution:

$$x^4 + 14x^3 + 33x^2 < 0 \tag{25}$$
$$x^2(x^2 + 14x + 33) < 0 \tag{26}$$
$$x^2(x+11)(x+3) < 0 \tag{27}$$

We get three values, $x = 0, x = -3, x = -11$ that make the left side 0. Placing these on number line and testing points, we get the following number line.

The given inequality is satisfied by x in $\boxed{(-11, -3)}$.

(b) $|5x+2| \leq 6$

Solution:

$$|5x+2| \leq 6 \quad (28)$$
$$-6 \leq 5x+2 \leq 6 \quad (29)$$
$$-8 \leq 5x \leq 4 \quad (30)$$
$$-\frac{8}{5} \leq x \leq \frac{4}{5} \quad (31)$$

$$\boxed{\left[-\frac{8}{5}, \frac{4}{5}\right]}$$

4. Find the value(s) for c such that the midpoint between $(3, c)$ and $(-4, 5)$ is $\left(-\frac{1}{2}, \frac{2}{3}\right)$. (4 pts)

Solution:

As $\left(-\frac{1}{2}, \frac{2}{3}\right)$ is the midpoint of $(3, c)$ and $(-4, 5)$, we have a relationship

$$\frac{c+5}{2} = \frac{2}{3} \quad (32)$$
$$6 \cdot \frac{c+5}{2} = 6 \cdot \frac{2}{3} \quad (33)$$
$$3c + 15 = 4 \quad (34)$$
$$3c + 15 - 315 = 4 - 15 \quad (35)$$
$$3c = -11 \quad (36)$$
$$c = \frac{-11}{3} \quad (37)$$

$$\boxed{c = -\frac{11}{3}}$$

5. Find the domain of the following functions. Express your answers in interval notation. (15 pts)

(a) $s(x) = \dfrac{8x^2-4}{x^2-5x}$

Solution:

$s(x)$ is a rational function, so its domain is all x values where the denominator does not equal 0. We start by factoring:

$$s(x) = \frac{4(2x^2-1)}{x(x-5)} \quad (38)$$

So we must have

$$x \neq 0 \quad (39)$$
$$x - 5 \neq 0 \quad (40)$$

So in interval notation, the domain is:

$$\boxed{(-\infty, 0) \cup (0, 5) \cup (5, \infty)} \quad (41)$$

(b) $h(x) = \dfrac{\sqrt{x-2}}{3x-9}$

Solution:

The domain of the function is determined by the x-values where the expression inside the square root to be positive or zero. We must also exclude all x values where the denominator is 0:

$$x - 2 \geq 0 \quad (42)$$
$$x \geq 2 \quad (43)$$
$$\quad (44)$$
$$3x - 9 \neq 0 \quad (45)$$
$$3x \neq 9 \quad (46)$$
$$x \neq 3 \quad (47)$$

So in interval notation:

$$\boxed{[2,3) \cup (3,\infty)} \quad (48)$$

(c) $f(r) = 5r^2 - 17r + 1$

Solution:

This function is a polynomial so has a domain of all real numbers:

$$\boxed{(-\infty,\infty)} \quad (49)$$

6. For $f(x) = 2 - 3x^2$ answer the following: (7 pts)

(a) Find $f(a)$

Solution:

$$f(a) = \boxed{2 - 3a^2} \quad (50)$$

(b) Find $f(a+h)$

Solution:

$$f(a+h) = 2 - 3(a+h)^2 \quad (51)$$
$$= 2 - 3(a^2 + 2ah + h^2) \quad (52)$$
$$= \boxed{2 - 3a^2 - 6ah - 3h^2} \quad (53)$$

(c) Find $\dfrac{f(a+h)-f(a)}{h}$ where h is a nonzero constant and simplify.

Solution:

$$\frac{f(a+h)-f(a)}{h} = \frac{(2-3a^2-6ah-3h^2)-(2-3a^2)}{h} \tag{54}$$
$$= \frac{2-3a^2-6ah-3h^2-2+3a^2}{h} \tag{55}$$
$$= \frac{-6ah-3h^2}{h} \tag{56}$$
$$= \frac{(-6a-3h)h}{h} \tag{57}$$
$$= \boxed{-6a-3h} \tag{58}$$

7. Find the inverse function of $g(x) = 2-5x^3$ (you may assume that $g(x)$ is one-to-one). (4 pts)

Solution: We start with

$$y = 2-5x^3 \tag{59}$$

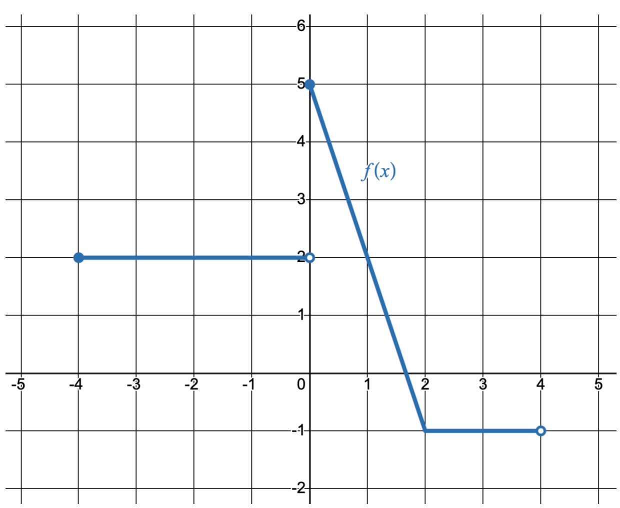
We interchange x and y and then solve for y:

$$x = 2-5y^3 \tag{60}$$
$$x-2 = -5y^3 \tag{61}$$
$$-x+2 = 5y^3 \tag{62}$$
$$\frac{2-x}{5} = y^3 \tag{63}$$
$$\sqrt[3]{\frac{2-x}{5}} = y \tag{64}$$

So our inverse function is

$$g^{-1}(x) = \boxed{\sqrt[3]{\frac{2-x}{5}}} \tag{65}$$

8. Answer the following for the given graph of a function $f(x)$. (15 pts)



(a) Find the domain of $f(x)$. Express your answer in interval notation.

Solution: $\boxed{[-4,4)}$

(b) Find the range of $f(x)$. Express your answer in interval notation.

Solution: $\boxed{[-1,5]}$

(c) Find $f(0)$

Solution: $\boxed{f(0)=5}$

(d) Find $(f\cdot f)(-3)$.

Solution: $f(-3)\cdot f(-3)=2\cdot 2=\boxed{4}$

(e) Find $(f\circ f)(1)$.

Solution: $f(f(1))=f(2)=\boxed{-1}$

(f) $f(x)$ is **not** one-to-one. Give a brief explanation as to why $f(x)$ is not one-to-one.

Solution: $f(x)$ fails the horizontal line test. In other words, there is a y-value that has two different x-values, x_1 and x_2, such that $f(x_1)=f(x_2)$. For example: $f(-3)=2=f(-2)$ but $-3\neq -2$.

(g) Find the equation of the line in the graph above on the restricted domain $[0,2]$.

Solution: The slope is

$$m=\frac{(-1-5)}{(2-0)}=\frac{-6}{2}=-3 \tag{66}$$

so the line is

$$y-5=-3x \tag{67}$$

$$y=\boxed{-3x+5} \tag{68}$$

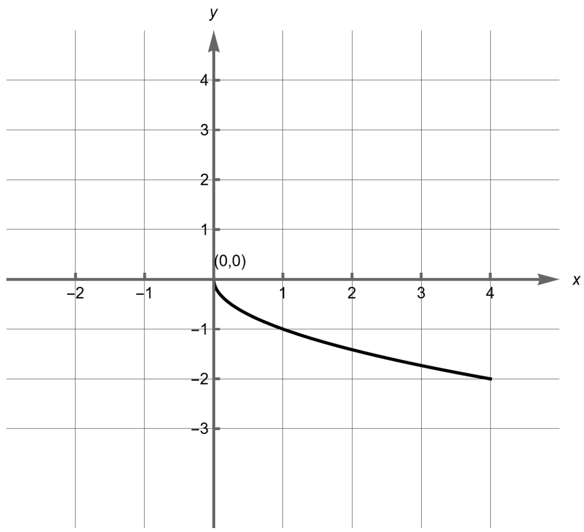
(h) Write down the piecewise-defined function whose graph is given above.

Solution:

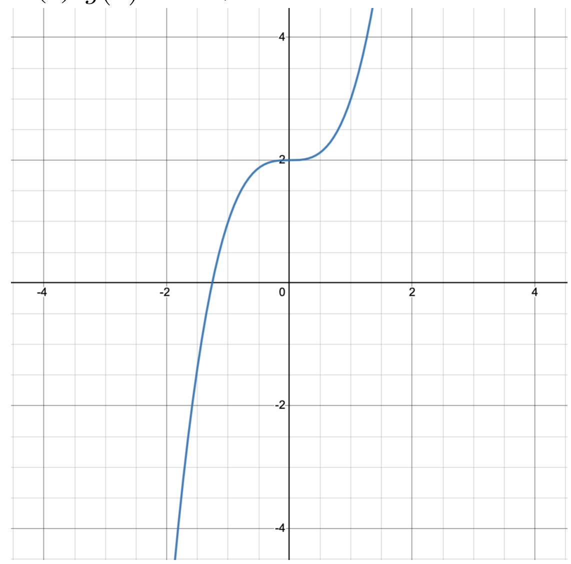
$$f(x)=\begin{cases}2 & \text{if } -4\le x<0\\ -3x+5 & \text{if } 0\le x\le 2\\ -1 & \text{if } 2<x<4\end{cases}$$

9. Sketch the shape of the graph of each of the following on the given set of axes. (15 pts)

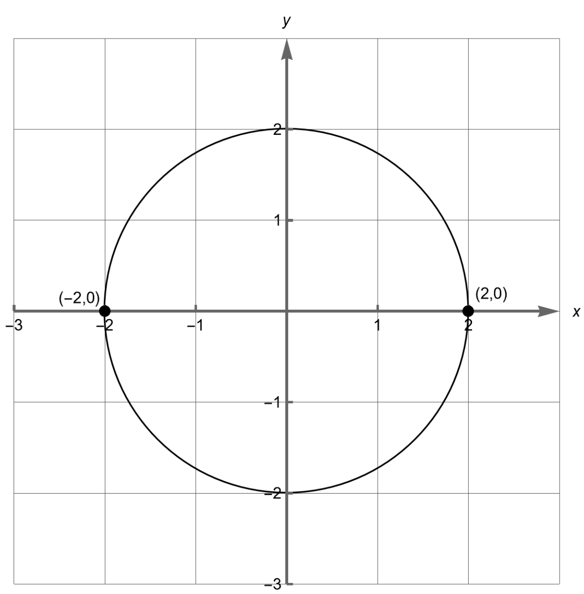
(a) $f(x) = -\sqrt{x}$



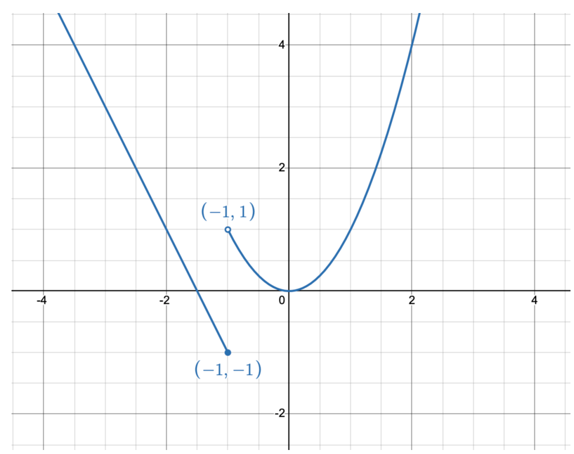
(b) $g(x) = x^3 + 2$



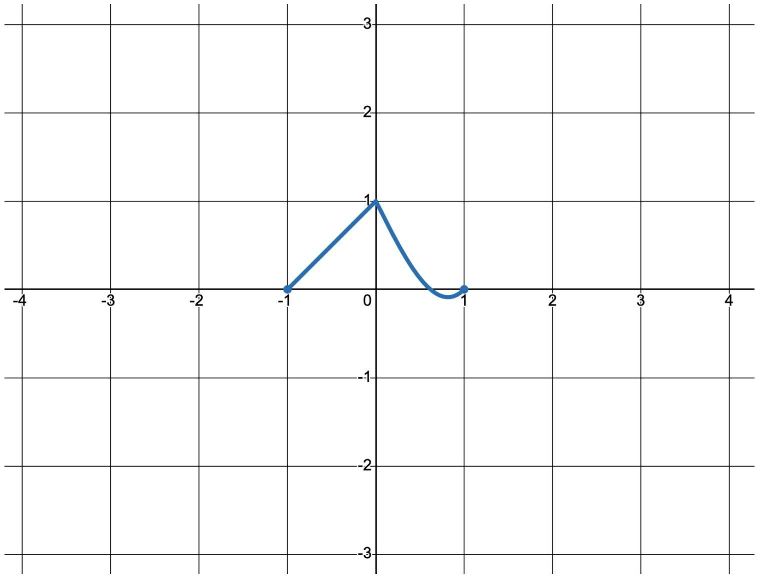
(c) $x^2 + y^2 = 4$



(d) $q(x) = \begin{cases} -2x - 3 & \text{if} \quad x \leq -1 \\ x^2 & \text{if} \quad x > -1 \end{cases}$

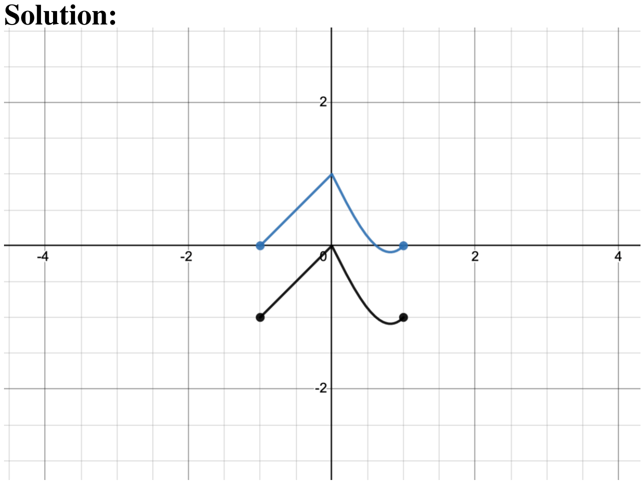


(e) For the function, $h(x)$, graphed below, answer the following:



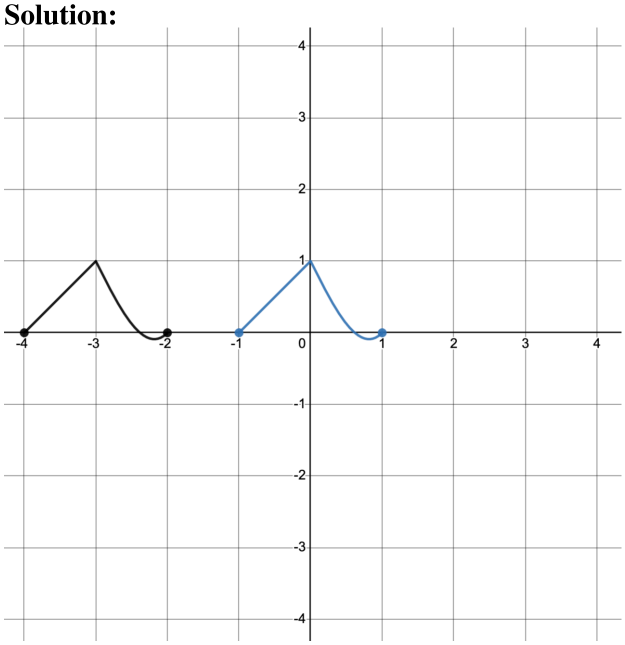
i. Use transformations to sketch the graph of $h(x) - 1$ **on the same graph above**.

Solution:



ii. Use transformations to sketch the graph of $h(x+3)$ **on the same graph above**.

Solution:



10. Is $f(x) = x^4 - |x| + 3$ odd, even, or neither? Justify your answer for credit. (4 pts)

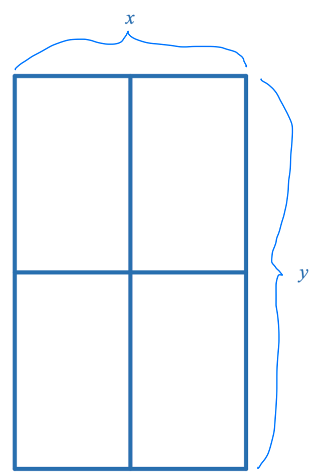
Solution:

We have

$$f(-x) = (-x)^4 - |-x| + 3 \quad (69)$$
$$= x^4 - |x| + 3 \quad (70)$$
$$= f(x) \quad (71)$$

Hence $f(x)$ is an $\boxed{\text{even}}$ function since $f(-x) = f(x)$ for all x.

11. A farmer is fencing in a field for her horses. The entire field is the shape of a rectangle with width x and length y. She is then going to add a length of fencing to split the field through the middle and another length of fencing to create four equal sized rectangular pens (the picture below illustrates the field after all fencing has been added). The farmer has a total of 930 ft of fencing available. Answer the following. (6 pts)



(a) Find an equation for the total area of the field as a function of width, x.

Solution:

The area of the rectangular field is given by $A = xy$. Now, the farmer has a total of 930 ft of fencing available, which means the total fencing used (including the inner fencing) is:

$$3x + 3y = 930 \quad (72)$$
$$x + y = 310 \quad (73)$$
$$y = 310 - x \quad (74)$$

Using this, we can eliminate y from the area equation and write the area as a function of x alone

$$A = xy \quad (75)$$
$$= x(310 - x) \quad (76)$$

Hence $\boxed{A(x) = -x^2 + 310x}$ ft^2

(b) Use the vertex formula to find the dimensions of the field that maximize the total area of the field.

Solution:

The above quadratic function opens down (since the coefficient of x^2 is negative). Hence it is maximized at the vertex. Using the vertex formula we get:

$$x = -\frac{310}{2(-1)} \tag{77}$$
$$= 155 \text{ ft} \tag{78}$$

Plugging this into the expression for y

$$y = 310 - x \tag{79}$$
$$= 155 \text{ ft} \tag{80}$$

Hence the dimensions of the field that maximize the total area are $\boxed{x = y = 155 \text{ ft}}$