1. (32 points) The following problems are not related.

   (a) Find the derivative of $y = \dfrac{x^2 - 4x + 8\sqrt{x}}{\sqrt{x}}$. (Simplify your final answer.)

   (b) Find $g'(x)$ where $g(x) = \dfrac{(5x - \tan(x))^4}{9x^2 - 4}$. (Please do **NOT** simplify your final answer.)

   (c) Find the slope of the tangent line at the point $(3, 2)$ of $xy^2 + y^3 = 20$.

Solution:

(a) We will simplify, and then use the power rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x^2 - 4x + 8\sqrt{x}}{\sqrt{x}}\right)\\ &= \frac{d}{dx}\left(\frac{x^2}{\sqrt{x}} - \frac{4x}{\sqrt{x}} + \frac{8\sqrt{x}}{\sqrt{x}}\right)\\ &= \frac{d}{dx}\left(x^{3/2} - 4x^{1/2} + 8\right)\\ &= \frac{3}{2}x^{1/2} - 2x^{-1/2}.\end{aligned}$$

(b) We will apply the quotient rule, and then use the chain rule:

$$\begin{aligned}g'(x) &= \frac{(9x^2 - 4)\frac{d}{dx}(5x - \tan(x))^4 - (5x - \tan(x))^4\frac{d}{dx}(9x^2 - 4)}{(9x^2 - 4)^2}\\ &= \frac{(9x^2 - 4)4(5x - \tan(x))^3(5 - \sec^2(x)) - (5x - \tan(x))^4(18x)}{(9x^2 - 4)^2}\end{aligned}$$

(c) First, we differentiate:

$$\begin{aligned}\frac{d}{dx}\left(xy^2 + y^3\right) &= \frac{d}{dx}(20)\\ 2xy\frac{dy}{dx} + y^2 + 3y^2\frac{dy}{dx} &= 0.\end{aligned}$$

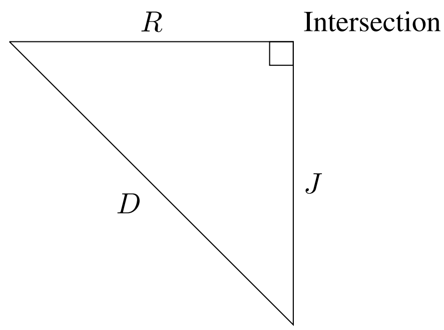
Now, we will plug in $(x, y) = (3, 2)$, and then solve for $\frac{dy}{dx}$:

$$\begin{aligned}2(3)(2)\frac{dy}{dx} + (2)^2 + 3(2)^2\frac{dy}{dx} &= 0\\ 24\frac{dy}{dx} &= -4\\ \frac{dy}{dx} &= -1/6.\end{aligned}$$

2. (12 points) Rumi is driving east **towards** an intersection at the same time Jinu is driving south **away** from the intersection. At the moment Rumi is 4 miles away from the intersection and driving at a speed of 50 miles per hour, Jinu is 6 miles away and driving at a speed of 30 miles per hour. At what rate is the distance between Rumi and Jinu changing at that moment?

Solution:

Let R represent the distance between Rumi and the intersection, J represent the distance between Jinu and the intersection, and let D be the distance between Rumi and Jinu. Note that these are all functions of time, t.



We want to know D' (or dD/dt) at that moment. By the Pythagorean Theorem, we have

$$J^2 + R^2 = D^2.$$

If we differentiate we have

$$2JJ' + 2RR' = 2DD'.$$

Solving for D', we obtain

$$D' = \frac{JJ' + RR'}{D}.$$

We are given that $R = 4$, $R' = -50$, $J = 6$, and $J' = 30$ at that moment. Using the Pythagorean theorem, we find

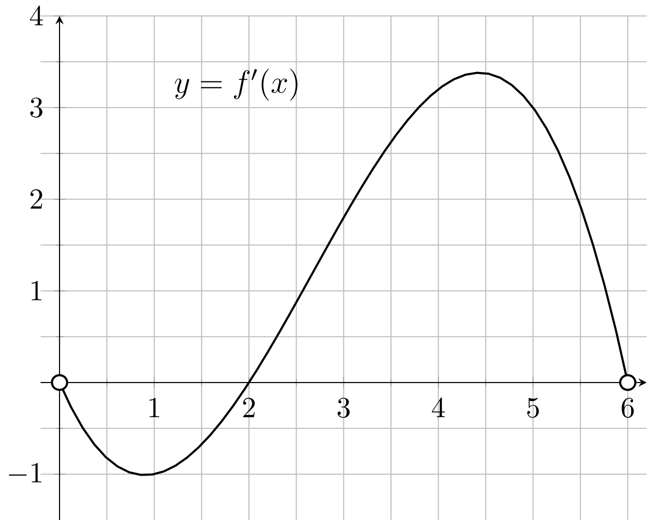
$$D = \sqrt{J^2 + R^2} = \sqrt{36 + 16} = \sqrt{52} = 2\sqrt{13}.$$

So, at that moment we have

$$D' = \frac{JJ' + RR'}{D} = \frac{6(30) + 4(-50)}{2\sqrt{13}} = -\frac{20}{2\sqrt{13}} = -\frac{10}{\sqrt{13}} \text{ miles per hour.}$$

That is, at that moment, the distance between Rumi and Jinu is decreasing at $10/\sqrt{13}$ miles per hour.

3. (12 points) Answer the following questions about the function $f(x)$, whose domain is $(0, 6)$. The graph of its derivative, $f'(x)$, is shown below. The derivative $f'(x)$ has a minimum value at $(1, -1)$ and a maximum value at $(4.4, 3.4)$. No justification is necessary for your answers.



(a) For what value(s) of x does $y = f(x)$ have horizontal tangents?

(b) On what open interval(s) does $y = f(x)$ have negative slope?

(c) On what open interval(s) is $f(x)$ continuous?

(d) Find an equation for the line tangent to $y = f(x)$ at $x = 5$ given $f(5) = -4$.

(e) Find the value of $\lim_{h\to 0} \dfrac{f(1+h) - f(1)}{h}$.

(f) On what open interval(s) is the second derivative, $f''(x)$, positive?

Solution:

(a) There is a horizontal tangent where the derivative $f'(x)$ equals 0 at $x = 2$.

(b) The function $f(x)$ has negative slope where the derivative $f'(x)$ has negative values on $(0, 2)$.

(c) If f is differentiable at a point, then f is continuous at a point, so f is continuous on $(0, 6)$.

(d) The tangent slope at $x = 5$ is $f'(5) = 3$, so an equation of the tangent line is $y = -4 + 3(x - 5) = 3x - 19$.

(e) The limit evaluates to $f'(1)$ which equals -1.

(f) The second derivative $f''(x)$ is positive where $f'(x)$ has positive slope on $(1, 4.4)$.

4. (22 points) Consider $g(x) = 4\cos x - 2\cos(2x)$.

(a) Determine all critical numbers of $g(x)$ in the interval $[0, \pi]$.

(b) Find the absolute maximum and absolute minimum values of $g(x)$ over the interval $[0, \pi]$. Clearly indicate the x-coordinates where these occur.

(c) Use the Intermediate Value Theorem to show that $g(x) = 2.5$ has at least one solution. First state the hypotheses of the intermediate value theorem and confirm that they are satisfied. Then determine an interval $[a, b]$ where the solution can be found.

Solution:

(a) We have

$$g'(x) = -4\sin x + 4\sin(2x) = -4\sin x + 8\sin x\cos x = -4\sin x(1 - 2\cos x).$$

If $g'(x) = 0$, then we have $\sin x = 0$ or $\cos x = 1/2$, which yields solutions

$$x = 0, \frac{\pi}{3}, \pi.$$

Since $g'(x)$ exists for all x, the values listed above are the critical numbers of g on this interval.

(b) We will evaluate g on the endpoints and critical numbers within our interval:

$$\begin{aligned} g(0) &= 4\cos 0 - 2\cos(2(0)) = 4 - 2 = 2 \\ g(\pi/3) &= 4\cos(\pi/3) - 2\cos(2(\pi/3)) = 4(1/2) - 2(-1/2) = 3 \\ g(\pi) &= 4\cos\pi - 2\cos(2(\pi)) = -4 - 2 = -6 \end{aligned}$$

So, the absolute maximum value is 3 at $x = \pi/3$ and the absolute minimum value is -6 at $x = \pi$.

(c) Note that $g(x)$ is continuous for all values because it is a combination of cosine functions, which are continuous for all values. Also, we see that 2.5 is between $g(0)$ and $g(\pi/3)$. By the the Intermediate Value Theorem, we know that $g(c) = 2.5$ for some c in $[0, \pi/3]$.

5. (22 points) Consider the function $f(x) = (x-9)^{2/3}$.

(a) i. Determine the linear approximation of $f(x)$ at $x = 17$.

ii. Use the linear approximation from (i) to approximate $f(17.2)$.

(b) Use the definition of the derivative to show that $f(x)$ is not differentiable at $x = 9$.

Solution:

(a) i. Recall that the linear approximation is just the tangent line. Note that

$$f(17) = 8^{2/3} = 2^2 = 4.$$

Also, we have $f'(x) = \frac{2}{3}(x-9)^{-1/3} = \frac{2}{3(x-9)^{1/3}}$, which means $f'(17) = \frac{2}{3(8)^{1/3}} = \frac{1}{3}$. So, the linear approximation is

$$L(x) = f(17) + f'(17)(x-17) = 4 + \frac{1}{3}(x-17).$$

ii.

$$f(17.2) \approx L(17.2) = 4 + \frac{1}{3}(17.2 - 17) = 4 + \frac{1}{3}\left(\frac{1}{5}\right) = \frac{61}{15}.$$

(b) We need to show that $f'(9)$ does not exist. We have

$$\begin{aligned} f'(9) &= \lim_{h\to 0} \frac{f(9+h) - f(9)}{h} \\ &= \lim_{h\to 0} \frac{h^{2/3} - 0}{h} \\ &= \lim_{h\to 0} \frac{1}{h^{1/3}} \end{aligned}$$

where

$$\lim_{h\to 0^+} \frac{1}{h^{1/3}} = \frac{1}{0^+} = \infty \text{ and } \lim_{h\to 0^-} \frac{1}{h^{1/3}} = \frac{1}{0^-} = -\infty.$$

So, $\lim_{h\to 0} \frac{f(9+h) - f(9)}{h}$ does not exist, which means $f(x)$ is not differentiable at $x = 9$.