1. (26 pts) Parts (a), (b), and (c) are not related to each other.

   (a) Is the function $f(x) = \dfrac{x^5}{x^2+5}$ odd, even, or neither? Justify your answer by using the definition of odd and/or even functions.

   Solution:

   $$f(-x) = \frac{(-x)^5}{(-x)^2+5} = \frac{-x^5}{x^2+5} = -\frac{x^5}{x^2+5} = -f(x)$$

   Since $f(-x) = f(x)$, $f(x)$ is an $\boxed{\text{odd function}}$

   (b) For $p(x) = \dfrac{1}{x^2-1}$ and $q(x) = \sqrt{1-x}$, identify the composite function $(p \circ q)(x)$ and its domain.

   Solution:

   $$(p \circ q)(x) = p(q(x)) = p\left(\sqrt{1-x}\right) = \frac{1}{(\sqrt{1-x})^2-1} = \frac{1}{(1-x)-1} = \boxed{-\frac{1}{x}}$$

   The domain of $q(x)$ is the set of all x values such that $1-x \geq 0$. So, domain of $q(x)$ is the set of all x values such that $x \leq 1$.

   The domain of $-\dfrac{1}{x}$ is the set of all x values such that $x \neq 0$.

   The domain of $(p \circ q)(x)$ is the set of all x values that satisfy both of the preceding two criteria. That is, x must satisfy $x \leq 1$ and $x \neq 0$.

   Therefore, the domain of $(p \circ q)(x)$ is $\boxed{(-\infty, 0) \cup (0, 1]}$

(c) The graph of $y = \sqrt{2x+1}$ is to be transformed in the following three steps, in the specified order:

i) Shifted horizontally by 4 units to the left
ii) Reflected across the y-axis
iii) Compressed vertically by a factor of 3

After each of the three transformations, what is the equation of the resulting graph? Only the final equation (the one that is obtained after the third transformation) needs to be simplified. Note that no actual graphing is required in this problem.

i. Equation of the graph after transformation (i):

Solution:

A horizontal shift by 4 units to the left is achieved by replacing x with $(x+4)$.

$$\boxed{y = \sqrt{2(x+4)+1}}$$

ii. Equation of the graph after transformations (i) and (ii):

Solution:

Reflection about the y-axis is achieved by replacing x with $-x$.

$$\boxed{y = \sqrt{2(-x+4)+1}}$$

iii. Equation of the graph after transformations (i), (ii), and (iii):

Solution:

A vertical compression by a factor of 3 is achieved by multiplying the function expression by $1/3$.

$y = \dfrac{1}{3}\sqrt{2(-x+4)+1}$ which simplifies to $\boxed{y = \dfrac{1}{3}\sqrt{9-2x}}$

2. (18 pts) Evaluate the following limits. If you use a named theorem, state the name as part of your solution.

(a) $\lim_{x\to 5} \dfrac{2-\sqrt{x-1}}{x^2-6x+5}$

Solution:

$$\begin{aligned}
\lim_{x\to 5} \frac{2-\sqrt{x-1}}{x^2-6x+5} &= \lim_{x\to 5} \frac{2-\sqrt{x-1}}{x^2-6x+5} \cdot \frac{2+\sqrt{x-1}}{2+\sqrt{x-1}} \\
&= \lim_{x\to 5} \frac{4-(x-1)}{(x-1)(x-5)(2+\sqrt{x-1})} \\
&= \lim_{x\to 5} \frac{5-x}{(x-1)(x-5)(2+\sqrt{x-1})} \\
&= \lim_{x\to 5} \frac{-1}{(x-1)(2+\sqrt{x-1})} \\
&= \frac{-1}{(5-1)(2+\sqrt{5-1})} \\
&= -\frac{1}{4\cdot(2+2)} = \boxed{-\frac{1}{16}}
\end{aligned}$$

(b) Suppose $g(x)$ is a function such that

$$-x^2+2x+2 \le g(x) \le x^3+x^2-5x+6$$

for all $x \ge 0$. Is there enough information to determine the value of $\lim_{x\to 1} g(x)$? If so, use the appropriate theorem to evaluate the limit.

Solution:

Since it is given that $-x^2+2x+2 \le g(x) \le x^3+x^2-5x+6$, the Squeeze Theorem might be applicable.

$$\lim_{x\to 1}(-x^2+2x+2) = -1^2+2\cdot 1+2 = 3$$
$$\lim_{x\to 1}(x^3+x^2-5x+6) = 1^3+1^2-5\cdot 1+6 = 3$$

Therefore, the **Squeeze Theorem** does apply, and it indicates that $\lim_{x\to 1} g(x) = \boxed{3}$

3. (19 pts) Consider the rational function $r(x) = \dfrac{x^3 - x^2 - 6x}{x^3 + 4x^2 + 4x}$.

(a) Identify all values of x, if any, for which $y = r(x)$ has a removable discontinuity. If none exist, clearly state "none". Support your answer by evaluating the appropriate limit(s).

Solution:

$$r(x) = \frac{x^3 - x^2 - 6x}{x^3 + 4x^2 + 4x} = \frac{x(x^2 - x - 6)}{x(x^2 + 4x + 4)} = \frac{x(x-3)(x+2)}{x(x+2)^2}$$

Therefore,

$$r(x) = \frac{x-3}{x+2} \text{ for } x \neq -2, 0$$

(The preceding simplified expression for $r(x)$ will be useful in part (b) as well.)

Since the simplified expression for $r(x)$ does not produce division by zero for $x = 0$, a two-sided limit can be evaluated.

$$\lim_{x \to 0} \frac{x-3}{x+2} = \frac{0-3}{0+2} = -\frac{3}{2}$$

Since $r(x)$ approaches a finite two-sided limit as x approaches 0, r has a removable discontinuity at $\boxed{x = 0}$

(b) Find the equation of each vertical asymptote of $y = r(x)$, if any exist. If none exist, clearly state "none". Support your answer by evaluating the appropriate limit(s).

Solution:

From part (a), we know that $r(x) = \dfrac{x-3}{x+2}$ for $x \neq -2, 0$.

Since this simplified function expression results in a zero denominator and a non-zero numerator for $x = -2$, one-sided limits must be evaluated.

$$\lim_{x \to -2^-} \frac{x-3}{x+2} \to \frac{-5}{0^-} = \infty$$

$$\lim_{x \to -2^+} \frac{x-3}{x+2} \to \frac{-5}{0^+} = -\infty$$

Either of the two preceding limits is sufficient to establish that r has a vertical asymptote at $\boxed{x = -2}$

4. (19 pts) Parts (a) and (b) are not related.

(a) For what value of a is the following function $u(x)$ continuous at $x = \pi/6$? Fully support your answer using the definition of continuity, which includes evaluating the appropriate limits.

$$u(x) = \begin{cases} 9ax^2 & , \quad x \le \pi/6 \\ \dfrac{\sin x}{x} & , \quad x > \pi/6 \end{cases}$$

Solution:

By definition, in order for $u(x)$ to be continuous at $x = \pi/6$, the following must be true:

$$\lim_{x\to\pi/6^-} u(x) = \lim_{x\to\pi/6^+} u(x) = u(\pi/6)$$

$$\lim_{x\to\pi/6^-} u(x) = \lim_{x\to\pi/6^-} 9ax^2 = 9a\left(\frac{\pi}{6}\right)^2 = \frac{a\pi^2}{4}$$

$$\lim_{x\to\pi/6^+} u(x) = \lim_{x\to\pi/6^+} \frac{\sin x}{x} = \frac{\sin(\pi/6)}{\pi/6} = \frac{6}{\pi}\cdot\frac{1}{2} = \frac{3}{\pi}$$

$$u(\pi/6) = 9a\left(\frac{\pi}{6}\right)^2 = \frac{a\pi^2}{4}$$

Therefore, we must have: $\dfrac{a\pi^2}{4} = \dfrac{3}{\pi}$.

$$a = \frac{3}{\pi}\cdot\frac{4}{\pi^2} = \boxed{\frac{12}{\pi^3}}$$

(b) Consider the function $y = v(x) = \dfrac{5x - 6}{x - 3}$.

Given that $v(0) = 2$ and $v(6) = 8$, can the Intermediate Value Theorem be used to show that $v(x) = 4$ for some value of x in the interval $(0, 6)$? Explain why or why not.

Solution:

Since $v(3)$ is undefined due to division by zero, v is not continuous at $x = 3$. Therefore, v is not continuous on the interval $[0, 6]$.

Since the continuity of v on $[0, 6]$ would be necessary in order to apply the Intermediate Value Theorem to this problem, the theorem can **not** be used to establish the existence of a value of x on $(0, 6)$ such that $v(x) = 4$.

5. (18 pts) Find all values of x in the interval $[0, 2\pi]$ that satisfy the following equation:

$$\sin(2x) = -\sqrt{3}\sin x$$

Solution:

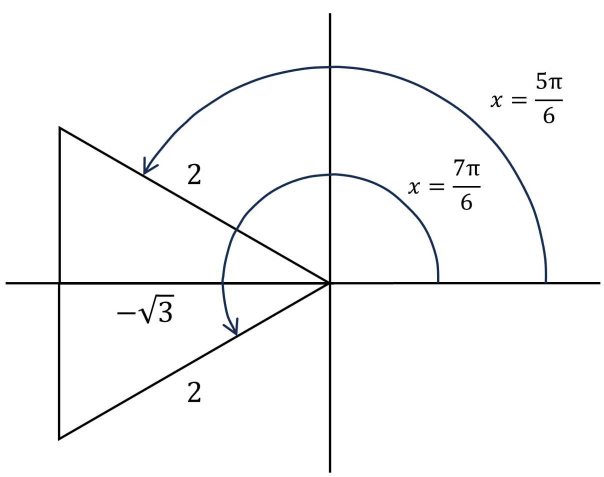
$$\begin{aligned} \sin(2x) &= -\sqrt{3}\sin x \\ 2\sin x\cos x &= -\sqrt{3}\sin x \\ 2\sin x\cos x + \sqrt{3}\sin x &= 0 \\ \sin x(2\cos x + \sqrt{3}) &= 0 \end{aligned}$$

The Zero Factor Theorem indicates that $\sin x = 0$ or $2\cos x + \sqrt{3} = 0$.

$\sin x = 0 \quad \Rightarrow \quad x = 0, \pi, 2\pi$

$2\cos x + \sqrt{3} = 0 \quad \Rightarrow \quad \cos x = -\dfrac{\sqrt{3}}{2}$

The following figure can assist in solving $\cos x = -\dfrac{\sqrt{3}}{2}$:



Both triangles in the preceding figure are associated with an angle whose cosine is $-\sqrt{3}/2$. The leg of length $\sqrt{3}$ and the hypotenuse of length 2 together imply that both triangles are special $30° - 60° - 90°$ right triangles. In each such triangle, the angle that is adjacent to the leg of length $\sqrt{3}$ is $\pi/6$ radians, which is the reference angle for both depicted triangles. Therefore, there are two solutions to the equation $\cos x = -\sqrt{3}/2$:

$x = \dfrac{5\pi}{6}, \dfrac{7\pi}{6}$

Combining the preceding result with the solutions to the equation $\sin x = 0$ found earlier yields the following five solutions to the original equation:

$$\boxed{x = 0, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, 2\pi}$$