1. (18 pts) The following questions are not related.

   (a) (6 points) Consider the rational function

   $$g(t) = \frac{t^3 - 2t - 4}{t^2(t-2)^3(t^2+t+1)^2}$$

   Write out the form for the partial fraction decomposition, but do not solve for the coefficients.

   (b) (6 points) Suppose that you have used the trig substitution $x = a\sin\theta$ (a is a positive constant) to integrate $y = f(x)$ and obtained

   $$\int f(x)\,dx = \frac{a^2\theta}{2} + \frac{a^2}{2}\sin(\theta)\cos(\theta) + C$$

   What is the final solution to the integral in terms of x?

   (c) (6 points) **True or false:** If f is a continuous, decreasing function on $[1,\infty)$ with $\lim_{x\to\infty} f(x) = 0$, then $\int_1^\infty f(x)\,dx$ is convergent. *You must justify your answer: if the statement is true, explain why. If it is false, you need to find an example that shows it is false.*

Solution:

(a) We first notice (using the quadratic formula and noting that the discrimant is negative) that t^2+t+1 is an irreducible quadratic. Hence:

$$\frac{t^3-2t-4}{t^2(t-2)^3(t^2+t+1)^2} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t-2} + \frac{D}{(t-2)^2} + \frac{E}{(t-2)^3} + \frac{Ft+G}{t^2+t+1} + \frac{Ht+I}{(t^2+t+1)^2}$$

(b) Use the reference triangle for $x = a\sin\theta$ to obtain

$$\int f(x)\,dx = \frac{a^2\theta}{2} + \frac{a^2}{2}\sin(\theta)\cos(\theta) + C = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2-x^2}}{2} + C$$

(c) False. Consider the function $f(x) = 1/x$ with domain $[1,\infty)$. On this domain, $f(x) = 1/x$ is continuous and decreasing and $\lim_{x\to\infty} f(x) = 0$. However, $\int_1^\infty 1/x\,dx$ diverges.

2. (34 pts) Evaluate the following integrals, and simplify your answers.

   (a) (10 pts) $\int \sin^3(\theta)\cos^2(\theta)\,d\theta$

   (b) (12 pts) $\int_7^{7\sqrt{2}} \frac{\sqrt{t^2-49}}{t}\,dt$

   (c) (12 pts) $\int_1^4 x^{1/2}\ln(x)\,dx$

Solution:

(a) Trig integral: Since there's an odd power of sine, we will pull off one power and convert the rest into cosine:

$$\int \sin^3\theta\cos^2\theta\,d\theta = \int \sin\theta\cdot\sin^2\theta\cos^2\theta\,d\theta$$

$$= \int \sin\theta\left(1-\cos^2\theta\right)\cos^2\theta\,d\theta$$
$$= \int \sin\theta\left(\cos^2\theta-\cos^4\theta\right)\,d\theta$$

Using the substitution $u=\cos\theta$, $du=-\sin\theta\,d\theta$ yields

$$= -\int \left(u^2-u^4\right)\,du$$
$$= -\left(\frac{1}{3}u^3-\frac{1}{5}u^5\right)+C$$
$$= \frac{1}{5}\cos^5\theta-\frac{1}{3}\cos^3\theta+C$$

(b) Trig substitution. Let $t=7\sec\theta$. Then $dt=7\sec\theta\tan\theta\,d\theta$ and $\sqrt{t^2-49}=\sqrt{49\sec^2\theta-49}=\sqrt{49\tan^2\theta}=7\tan\theta$. We can convert the limits $x=7$ and $x=7\sqrt{2}$ to $\theta=0$ and $\theta=\pi/4$, respectively.

$$\int_7^{7\sqrt{2}}\frac{\sqrt{t^2-49}}{t}\,dt=\int_0^{\pi/4}\frac{7\tan\theta}{7\sec\theta}(7\sec\theta\tan\theta)\,d\theta=\int_0^{\pi/4}7\tan^2\theta\,d\theta$$
$$=\int_0^{\pi/4}7\left(\sec^2\theta-1\right)\,d\theta=\Big[7(\tan\theta-\theta)\Big]_0^{\pi/4}$$
$$=\boxed{7\left(1-\frac{\pi}{4}\right)}$$

(c) Integration by parts: Let $u=\ln x$, $du=(1/x)\,dx$, $dv=x^{1/2}dx$ and $v=(2/3)x^{3/2}$. Then:

$$\begin{aligned}\int_1^4 x^{1/2}\ln(x)\,dx &= \left[\frac{2}{3}x^{3/2}\ln x\right]_1^4+\int_1^4(2/3)x^{1/2}\,dx\\ &= \frac{2}{3}(4)^{3/2}\ln 4-\left[\frac{2}{3}\cdot\frac{2}{3}x^{3/2}\right]_1^4\\ &= \frac{16}{3}\ln 4-\frac{4}{9}\left(4^{3/2}-1\right)\\ &= \boxed{\frac{16}{3}\ln 4-\frac{28}{9}}\end{aligned}$$

3. (24 pts) Determine whether the following integrals are convergent or divergent. Explain your reasoning fully for each integral. If an integral can be evaluated, please do so. (**If you use the Comparison Test, state this and evaluate the integral that you are using for comparison.**)

(a) $\displaystyle\int_2^\infty \frac{1}{(u-1)(u+3)}du$

(b) $\displaystyle\int_0^\infty \frac{1}{x^2}\,dx$

(c) $\displaystyle\int_3^\infty \frac{x^2}{x^{5/2}-2}\,dx$

Solution:

(a) The integrand has asymptotes at $u = 1$ and $u = -3$, neither of which are within the domain of integration $[2, \infty)$. Hence, we can evaluate this directly using partial fraction decomposition. To this end, writing

$$\frac{1}{(u-1)(u+3)} = \frac{A}{u-1} + \frac{B}{u+3}$$

yields the relation

$$1 = A(u+3) + B(u-1)$$

after clearing the denominator on the left-hand side. Plugging in $u = 1$ and $u = -3$ and solving for the constants gives $A = 1/4$ and $B = -1/4$. Alternatively, grouping terms and equating constants gives the system

$$\begin{aligned} A + B &= 0 \\ 3A - B &= 1, \end{aligned}$$

which has the same solution. With these constants, we can evaluate the integral

$$\begin{aligned} \int_2^\infty \frac{1}{(u-1)(u+3)} du &= \frac{1}{4} \int_2^\infty \left(\frac{1}{u-1} - \frac{1}{u+3} \right) du \\ &= \frac{1}{4} \lim_{t\to\infty} \int_2^t \left(\frac{1}{u-1} - \frac{1}{u+3} \right) du \\ &= \frac{1}{4} \lim_{t\to\infty} \left(\ln|u-1| - \ln|u+3| \right)\Big|_2^t \end{aligned}$$

since both quantities $u-1$ and $u+3$ are positive on $[2, \infty)$, it's OK to drop the absolute-value bars:

$$\begin{aligned} &= \frac{1}{4} \lim_{t\to\infty} \ln\left(\frac{u-1}{u+3} \right)\Big|_2^t \\ &= \frac{1}{4} \lim_{t\to\infty} \left[\ln\left(\frac{t-1}{t+3} \right) - \ln(1/5) \right] \\ &= \frac{1}{4}(\ln(1) - \ln(1/5)) \\ &\boxed{= -\frac{1}{4}\ln(1/5) = \frac{1}{4}\ln(5) = \ln\left(5^{1/4}\right) = \ln\left(\sqrt[4]{5}\right).} \end{aligned}$$

(b) For $\displaystyle\int_0^\infty \frac{1}{x^2}\,dx$ to converge, both of the integrals $\displaystyle\lim_{a\to 0+} \int_a^1 \frac{1}{x^2}\,dx$ and $\displaystyle\lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$ must converge. We see that:

$$\lim_{a\to 0+} \int_a^1 \frac{1}{x^2}\,dx = \lim_{a\to 0+} \left[\frac{-1}{x} \right]_a^1 = \lim_{a\to 0+} \left[-1 + \frac{1}{a} \right]$$

which diverges. Therefore, the original integral, $\displaystyle\int_0^\infty \frac{1}{x^2}\,dx$ $\boxed{\textit{diverges}}$. (Note, the second integral, $\displaystyle\lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$ does converge. But, once we found that the first integral diverged, we didn't need to look at the second integral.)

(c) This integral seems quite tricky to evaluate directly. So let's try to use the Comparison Theorem. Since $x \geq 3$, we have that

$$\frac{x^2}{x^{5/2} - 2} \geq \frac{x^2}{x^{5/2}} = \frac{1}{\sqrt{x}}$$

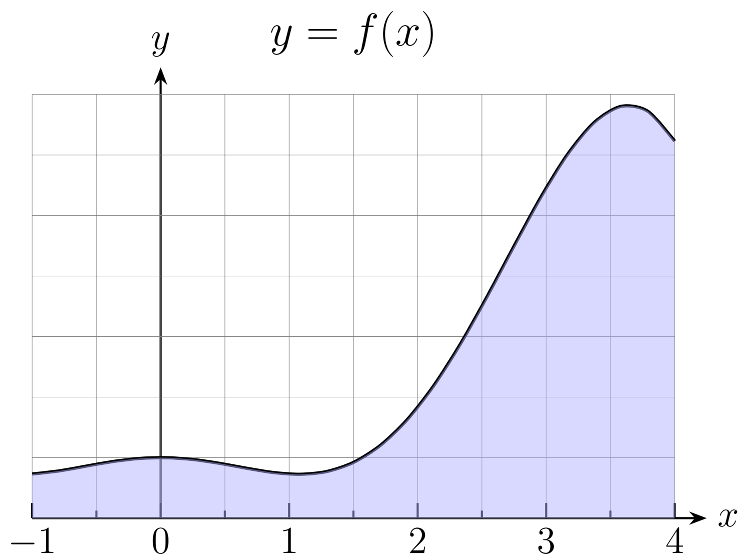
where we have made the denominator bigger (and hence a smaller fraction). Further,

$$\int_3^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty}\int_3^t x^{-1/2}\,dx = \lim_{t\to\infty} 2\sqrt{t} - 2\sqrt{3} = \infty$$

meaning our comparison diverges (this integral is also a divergent p-integral with $p = 1/2 \leq 1$). Then, by the Comparison Theorem, $\int_3^\infty \frac{x^2}{x^{5/2}-2}\,dx$ $\boxed{\text{diverges.}}$

4. (24 pts) All parts of this problem refer to the same function f, whose graph is below. The left, right, trapezoidal, and midpoint rule approximations were used to estimate $\displaystyle\int_{-1}^{4} f(x)\,dx$.

For $n = 2$ subintervals, the estimates were 8.25, 22.00, 27.32, and 35.75.



(a) Which rule produced which estimate? (Write down L_2, R_2, T_2, and M_2 and match each to its estimate.)

(b) For each of the four estimates, state whether it is an underestimate or an overestimate of the true value of $\displaystyle\int_{-1}^{4} f(x)\,dx$.

(c) Suppose the number of subintervals in the trapezoidal approximation is increased to $n = 50$, and suppose $f''(x) = -2\cos x + x^2\cos x + 4x\sin x$. Estimate the error in T_{50}. You do not need to simplify the values in your numerator and denominator.

(d) Now suppose you want to estimate the value of the integral with error less than 10^{-6} using the trapezoidal rule. What value of n should you use? (Use $f''(x)$ from part (c). You should solve for n but you do not need to simplify your answer further.)

Solution:

(a) Drawing rectangles and trapezoids for L_2, R_2, T_2, and M_2, we find that

$$\boxed{L_2 = 8.25,\ T_2 = 22.00,\ M_2 = 27.32,\ R_2 = 35.75.}$$

(b) From the graph, L_2 and T_2 underestimate the area while M_2 and R_2 overestimate the area.

(c) To start, we need to find K for the error formula

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}$$

Since $0 \leq |\cos x| \leq 1$ and $0 \leq |\sin x| \leq 1$, we have that

$$\begin{aligned} |f''(x)| &= \left|-2\cos x + x^2 \cos x + 4x \sin x\right| \\ &\leq |-2||\cos x| + |x^2||\cos x| + |4x|\sin x| \\ &\leq 2 + |x^2| + 4|x| \\ &\leq 2 + 4^2 + 4 \cdot 4 \\ &= 34 = K. \end{aligned}$$

Plugging everything in, we find that

$$|E_T| \leq \frac{34(4-(-1))^3}{12 \cdot 50^2} = \boxed{\frac{34 \cdot 5^3}{12 \cdot 50^2}.}$$

(d) Using $K = 34$ from part (c), we have that

$$|E_T| \leq \frac{34 \cdot 5^3}{12n^2} < 10^{-6} \implies \frac{34 \cdot 5^3}{12 \cdot 10^{-6}} < n^2 \implies \boxed{n > \sqrt{\frac{34 \cdot 5^3}{12 \cdot 10^{-6}}}.}$$