1. (30 pts) Triangle ABC has vertices at $A(2,1,0)$, $B(6,-2,0)$, and $C(7,3,0)$.

   (a) Find a vector of length $\frac{1}{5}$ in the same direction as $\overrightarrow{AB}$.

   (b) Find an equation in symmetric form for the line passing through A and B.

   (c) Find all unit vectors orthogonal to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

   (d) Find the area of $\triangle ABC$.

   (e) An altitude is drawn from vertex C to side $\overline{AB}$. Find a unit vector parallel to this altitude.

Solution:

(a) The vector $\mathbf{AB} = \langle 4,-3,0\rangle$ has a length of $\sqrt{4^2+3^2} = 5$. Dividing the vector by 25 gives $\boxed{\frac{1}{25}\langle 4,-3,0\rangle} = \boxed{\left\langle \frac{4}{25}, -\frac{3}{25}, 0\right\rangle}$ which has a length of $\frac{1}{5}$.

(b) The line will be parallel to $\mathbf{AB} = \langle 4,-3,0\rangle$ and will pass through point $A(2,1,0)$, so an equation of the line in vector form is

$$\mathbf{r}(t) = \langle 2,1,0\rangle + t\langle 4,-3,0\rangle = \langle 2+4t, 1-3t, 0\rangle$$

and in symmetric form, the line is

$$\boxed{\frac{x-2}{4} = \frac{y-1}{-3}, z = 0}.$$

Other representations for the line are possible.

(c) Both vectors $\mathbf{AB}$ and $\mathbf{AC}$ lie in the xy-plane, so the unit vectors orthogonal to them are $\boxed{\mathbf{k}, -\mathbf{k}}$, or $\boxed{\langle 0,0,1\rangle, \langle 0,0,-1\rangle}$.

Alternate Solution: Calculate $\mathbf{AB} \times \mathbf{AC} = 23\,\mathbf{k}$, then divide by the magnitude of 23.

(d) The area equals half the area of the parallelogram formed by vectors $\mathbf{AB} = \langle 4,-3,0\rangle$ and $\mathbf{AC} = \langle 5,2,0\rangle$:

$$\frac{1}{2}\left|\mathbf{AB}\times\mathbf{AC}\right| = \frac{1}{2}\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3 & 0 \\ 5 & 2 & 0 \end{matrix}\right\| = \frac{1}{2}\left|23\mathbf{k}\right| = \boxed{\frac{23}{2}}.$$

(e) Note that the triangle is in the xy-plane. Side $\overline{AB}$ has a slope of $-\frac{3}{4}$ in the plane. The altitude is perpendicular to $\overline{AB}$ so its slope is $\frac{4}{3}$, the negative reciprocal. A vector with this slope is $\langle 3,4,0\rangle$. Dividing by its length of 5 gives the unit vector $\boxed{\frac{1}{5}\langle 3,4,0\rangle} = \boxed{\langle \frac{3}{5}, \frac{4}{5}, 0\rangle}$. Another unit vector is $\langle -\frac{3}{5}, -\frac{4}{5}, 0\rangle$.

2. (30 pts) Axis Ant begins on the ground and walks along a straight path

$$\mathbf{r}(t) = t\,\mathbf{i} + (-1+2t)\,\mathbf{j} + 2t\,\mathbf{k}, \quad t \geq 0.$$

When it reaches point $P(1,1,2)$, it sees Butter Fly hovering at point Q.

(a) Given vector $\overrightarrow{PQ} = \langle 3,-2,6\rangle$, how high above the ground is Butter Fly?

(b) What is the angle between $\overrightarrow{PQ}$ and the path?

(c) Find an equation for the plane containing both the path and point Q. Fully simplify your answer.

(d) Butter Fly remains at Q while Axis Ant continues walking along the path until it reaches R, the point on the path closest to Q. What is the distance Axis Ant travels from P to R?

Solution:

(a) Point Q has coordinates $(1,1,2)+(3,-2,6)=(4,-1,8)$, so Butter Fly is $\boxed{\text{8 units}}$ above the ground ($z=0$).

(b) The path $\mathbf{r}(t) = \langle 0,-1,0\rangle + t\langle 1,2,2\rangle$ is parallel to $\mathbf{v} = \langle 1,2,2\rangle$. Given $|\mathbf{PQ}| = \sqrt{3^2+2^2+6^2} = 7$ and $|\mathbf{v}| = \sqrt{1^2+2^2+2^2} = 3$, the angle between $\mathbf{PQ}$ and the path is

$$\theta = \cos^{-1}\left(\frac{\mathbf{PQ}\cdot\mathbf{v}}{|\mathbf{PQ}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{\langle 3,-2,6\rangle\cdot\langle 1,2,2\rangle}{7\cdot 3}\right) = \boxed{\cos^{-1}\left(\frac{11}{21}\right)}.$$

(c) A vector normal to the plane is

$$\mathbf{n} = \mathbf{v}\times\mathbf{PQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 3 & -2 & 6 \end{vmatrix} = 16\mathbf{i} - 8\mathbf{k}.$$

Using point $P(1,1,2)$, an equation of the plane is

$$16(x-1) - 8(z-2) = 0 \implies 2(x-1) - (z-2) = 0 \implies \boxed{2x - z = 0}.$$

(d) The distance PR equals

$$\text{comp}_{\mathbf{v}}\mathbf{PQ} = \frac{\mathbf{v}\cdot\mathbf{PQ}}{|\mathbf{v}|} = \boxed{\frac{11}{3}}.$$

Alternate solution: The distance from point Q to the line is

$$QR = d_{\text{line}} = \frac{|\mathbf{PQ}\times\mathbf{v}|}{|\mathbf{v}|} = \frac{|\langle 16,0,-8\rangle|}{3} = \frac{|8\langle 2,0,-1\rangle|}{3} = \frac{8\sqrt{5}}{3}.$$

By the Pythagorean Theorem,

$$PR = \sqrt{|\mathbf{PQ}|^2 - QR^2} = \sqrt{49 - \frac{64\cdot 5}{9}} = \sqrt{\frac{121}{9}} = \frac{11}{3}.$$

3. (20 pts) Consider the surface $4y^2 - z^2 - 4x - 24y - 4z + 32 = 0$.

(a) Write the equation in standard form.

(b) Identify the surface by classifying the type of curves in the $x = 0$, $y = 0$, and $z = 0$ traces.

(c) Suppose the surface is intersected with $z = 2y - 2$. Find a vector function representing the curve of intersection.

Solution:

(a)

$$
\begin{aligned}
4y^2 - z^2 - 4x - 24y - 4z + 32 &= 0 \\
-4x + 4\left(y^2 - 6y\right) - \left(z^2 + 4z\right) + 32 &= 0 \\
\boxed{-4x + 4(y-3)^2 - (z+2)^2 = 0} \\
\boxed{-x + (y-3)^2 - \frac{(z+2)^2}{4} = 0}
\end{aligned}
$$

(b) The $x = 0$ trace is $4(y-3)^2 - (z+2)^2 = 0$ which corresponds to $\boxed{\text{two lines}}$:

$$4(y-3)^2 = (z+2)^2 \implies 2|y-3| = z+2 \implies z = 2y - 8 \text{ and } z = -2y + 4.$$

The $y = 0$ trace is $4x + (z+2)^2 = 36$ which is a $\boxed{\text{parabola}}$.

The $z = 0$ trace is $-4x + 4(y-3)^2 = 4$ which is a $\boxed{\text{parabola}}$.

The surface is a $\boxed{\text{hyperbolic paraboloid}}$.

(c) Let $y = t$ and $z = 2t - 2$. Then solving for x gives

$$-4x + 4(t-3)^2 - (2t)^2 = 0 \implies -x + (t-3)^2 - t^2 = 0 \implies x = -6t + 9.$$

Therefore a vector function for the curve of intersection is $\boxed{\mathbf{r}(t) = \langle -6t + 9, t, 2t - 2\rangle}$. Other representations for the curve are possible.

4. (20 pts) Beta Bug leaves home at time $t = 0$ and travels along the path

$$\mathbf{r}(t) = t\sin(t)\,\mathbf{i} + \cos(t)\,\mathbf{j} + (12\pi - 3t)\,\mathbf{k}.$$

(a) At $t = \pi$, how far is Beta Bug from home?

(b) At that moment, the bug decides to leave the path and travel in a straight line in the direction of the tangent. Find a vector equation for the line. (The parameter in the equation may begin at 0.)

(c) Find the coordinates of the point where Beta Bug will land on the ground ($z = 0$).

Solution:

(a) Beta Bug's home is located at $\mathbf{r}(0) = \langle 0, 1, 12\pi \rangle$. At $t = \pi$, the bug is at $\mathbf{r}(\pi) = \langle 0, -1, 9\pi \rangle$. The distance to home is

$$|\mathbf{r}(\pi) - \mathbf{r}(0)| = |\langle 0, -2, -3\pi \rangle| = \boxed{\sqrt{4 + 9\pi^2}}.$$

(b) Find the tangent vector at $t = \pi$.

$$\begin{aligned} \mathbf{r}'(t) &= \langle t\cos(t) + \sin(t), -\sin(t), -3 \rangle \\ \mathbf{r}'(\pi) &= \langle -\pi, 0, -3 \rangle \end{aligned}$$

An equation for the line is

$$\boxed{\mathbf{s}(t) = \langle 0, -1, 9\pi \rangle + t\langle -\pi, 0, -3 \rangle}$$
$$\boxed{\mathbf{s}(t) = \langle -\pi t, -1, 9\pi - 3t \rangle, \quad t \geq 0}.$$

(c) The path will reach $z = 0$ when $9\pi - 3t = 0$ after an additional $t = 3\pi$ time units. Beta Bug will land on the ground when $\mathbf{s}(3\pi) = \langle -3\pi^2, -1, 0 \rangle$ at the point $\boxed{(-3\pi^2, -1, 0)}$.