1. [2360/091725 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

   (a) $x^2 - 2xy - y^2 = 4$ is the general solution of $(x+y)y' = x - y$.

   (b) The differential equation $\dfrac{\mathrm{d}z}{\mathrm{d}t} - z^3 + 2z = 2$ has three equilibrium solutions that are rational numbers.

   (c) $(y'')^3 + 7y' - y = e^t$ is a second order nonlinear differential equation.

   (d) For all solutions, $y(t)$, of the differential equation $y' + y^2 = -1$, $\lim_{t\to\infty} y(t) = -1$.

   (e) If $x_1(t)$ and $x_2(t)$ are solutions of $\dfrac{1}{t-1}x'' + (t+1)x' + x = 0$, then $7x_1 - 11x_2$ is also a solution.

   SOLUTION:

   (a) **FALSE** It contains no parameters and consequently cannot be a general solution. Note that implicit differentiation shows that it is a particular solution.

   (b) **FALSE** The equilibrium solutions are roots of $z^3 - 2z + 2 = 0$. The Rational Roots Theorem states that the only possible rational roots are $\pm 1, \pm 2$. Using either synthetic division or direct evaluation shows that none of these are roots and thus the differential equation has no rational equilibrium solutions.

   (c) **TRUE** The highest derivative is the second derivative and since that term is cubed the equation is nonlinear.

   (d) **FALSE** Since y' is everywhere negative, all solutions approach $-\infty$ as $t \to \infty$.

   (e) **TRUE** The equation is linear and homogeneous so the Superposition Principle applies.

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2. [2360/091725 (14 pts)] Consider the differential equation $\dfrac{\mathrm{d}x}{\mathrm{d}t} = x^6 - x^4$.

   (a) (3 pts) For each of the following pairs giving various classifications of differential equations, in your solutions write the correct one that the equation satisfies: linear/nonlinear; separable/nonseparable; autonomous/nonautonomous

   (b) (5 pts) For the initial condition $x(1) = 2$ and step size $1/16$, find the approximation to the true solution after one step of Euler's method. Would your answer change if the initial condition were $x(t_0) = 2$ for any $t_0 \neq 1$? Briefly explain.

   (c) (6 pts) Find all equilibrium solutions and determine their stability.

   SOLUTION:

   (a) nonlinear, separable, autonomous

   (b) Euler's formula for this is $x_{n+1} = x_n + h\left(x_n^6 - x_n^4\right)$. Thus

   $$x_1 = x_0 + h\left(x_0^6 - x_0^4\right) = 2 + \frac{1}{2^4}\left(2^6 - 2^4\right) = 2 + (4-1) = 5$$

   No, the approximation would be the same since the independent variable is not part of the computation.

   (c) $x^6 - x^4 = x^4(x^2-1) = x^4(x+1)(x-1) \implies$ equilibrium solutions are $x = -1, 0, 1$

   $$\begin{aligned} x < -1 &\implies x' > 0 \\ -1 < x < 0 &\implies x' < 0 \\ 0 < x < 1 &\implies x' < 0 \\ 1 < x &\implies x' > 0 \end{aligned}$$

   Thus, $x = -1$ is stable, $x = 0$ is semistable, $x = 1$ is unstable.

3. [2360/091725 (17 pts)] Consider the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = \frac{1}{x}$, $x > 0$.

(a) (3 pts) Find the isocline along which the slope of the solution is 1.

(b) (10 pts) Use the integrating factor method to find the general solution. No points for using any other method.

(c) (4 pts) Find the solution passing through the point $(1, 1)$.

SOLUTION:

(a) Rewrite the equation as $y' = \frac{y+1}{x^2}$. The isocline is then given by

$$\frac{y+1}{x^2} = 1 \implies y = x^2 - 1$$

(b) Rewrite the equation as $y' - \frac{y}{x^2} = \frac{1}{x^2}$. Then $\int p(x)\,\mathrm{d}x = \int -x^{-2}\,\mathrm{d}x = x^{-1} \implies \mu = e^{1/x}$ and

$$\left(e^{1/x}y\right)' = \frac{e^{1/x}}{x^2}$$

$$e^{1/x}y = \int \left(e^{1/x}y\right)'\,\mathrm{d}x = \int \frac{e^{1/x}}{x^2}\,\mathrm{d}x \overset{u=1/x}{=} -e^{1/x} + C$$

$$y = -1 + Ce^{-1/x}$$

(c) Apply the initial condition, $y(1) = 1$.

$$1 = -1 + Ce^{-1} \implies C = 2e \implies y = \frac{2e}{e^{1/x}} - 1$$

4. [2360/091725 (18 pts)] Using Newton's second law, the velocity, u, of an object with mass m experiencing a gravitational force of mg and air resistance proportional to the velocity is given by the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = g - \frac{k}{m}u$$

where $k > 0$ is a constant and m and g are also positive constants.

(a) (5 pts) For any initial condition $u(t_0) = u_0$, what conclusions, if any, can be drawn from Picard's Theorem? Justify your answer.

(b) (10 pts) Use variation of parameters (Euler-Lagrange Two Step Method) to find the general solution to the equation. No points for using any other method.

(c) (3 pts) Does the solution possess a steady state portion? If so, what is it? If not, explain why not.

SOLUTION:

(a) $f(t, u) = g - \frac{k}{m}u$ and $f_u(t, u) = -\frac{k}{m}$ are both continuous for all t, u. Thus, Picard's Theorem guarantees that a unique solution exists for any initial condition.

(b) Solve the associated homogeneous problem first using separation of variables.

$$\int \frac{\mathrm{d}u_h}{u_h} = -\int \frac{k}{m}\,\mathrm{d}t$$

$$\ln|u_h| = -\frac{k}{m}t + C_1$$

$$|u_h| = e^{-kt/m + C_1}$$

$$u_h = Ce^{-kt/m}$$

Now let $u_p = v(t)e^{-kt/m}$ and substitute into the nonhomogeneous equation.

$$-\frac{k}{m}ve^{-kt/m} + v'e^{-kt/m} + \frac{k}{m}ve^{-kt/m} = g$$

$$v' = ge^{kt/m}$$

$$\int v'\,\mathrm{d}t = \int ge^{kt/m}\,\mathrm{d}t$$

$$v(t) = \frac{mg}{k}e^{kt/m} \implies u_p = \frac{mg}{k}e^{kt/m}e^{-kt/m} = \frac{mg}{k}$$

Thus $u = u_h + u_p = Ce^{-kt/m} + \dfrac{mg}{k}$.

(c) Yes, the steady state solution is mg/k (it is bounded and remains as $t \to \infty$). As noted in WebAssign, this is called the *terminal velocity*.

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5. [2360/091725 (8 pts)] A 1000-liter holding tank is initially half full of a well-mixed solution containing 30 kilograms (kg) of Magic Potion X. When time starts ($t = 0$), the solution flows out of the tank at a rate of 5 liters (L) per hour (hr). Simultaneously, the flow into the tank is 10 L/hr. A wizard sitting above the tank is putting $(2 + \cos t)$ kg/L of Magic Potion X into the liquid flowing into the tank. Letting $x(t)$ be the amount of Magic Potion X in the tank at time t, write down, but **do not solve**, the initial value problem (IVP) whose solution is $x(t)$.

SOLUTION:

Since the flow rates in to and out of the tank differ, the volume of solution in the tank will vary with time.

$$\frac{\mathbf{d}V}{\mathbf{d}t} = \text{inflow rate} - \text{outflow rate} = 10 - 5 = 5, V(0) = 500 \implies V(t) = 500 + 5t$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \text{mass rate in} - \text{mass rate out} = \left(2 + \cos t\,\frac{\text{kg}}{\text{L}}\right)\left(10\,\frac{\text{L}}{\text{hr}}\right) - \left(\frac{x}{500+5t}\,\frac{\text{kg}}{\text{L}}\right)\left(5\frac{\text{L}}{\text{hr}}\right)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{x}{100+t} = 20 + 10\cos t,\ x(0) = 30$$

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6. [2360/091725 (18 pts)] Consider the system of differential equations

$$x' = x - xy$$

$$y' = y + xy$$

(a) (4 pts) Find the h nullclines, if any exist.

(b) (4 pts) Find the v nullclines, if any exist.

(c) (4 pts) Find the equilibrium solutions, if any exist.

(d) (3 pts) Draw a phase plane in your bluebook with an arrow indicating the direction of the trajectory at the point $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

(e) (3 pts) If $x(0) = -1$ and $y(0) = 1$, find $\lim\limits_{t\to\infty} x(t)$ and $\lim\limits_{t\to\infty} y(t)$.

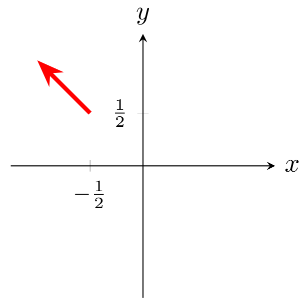
SOLUTION:

(a) h nullclines occur when $y' = y(1 + x) = 0$ so $y = 0$ and $x = -1$.

(b) v nullclines occur when $x' = x(1 - y) = 0$ so $x = 0$ and $y = 1$.

(c) Equilibrium solutions occur when both x' and y' evaluate to 0 or where the nullclines intersect. This occurs at $(0, 0)$ and $(-1, 1)$

(d)



(e) Since $(-1, 1)$ is an equilibrium solution, $x' = y' = 0 \implies \lim_{t\to\infty} x(t) = -1$ and $\lim_{t\to\infty} y(t) = 1$.

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7. [2360/091725 (15 pts)] In the homework there were several examples of using substitutions (changes of variables) to solve differential equations. Using the substitution $v = y/t$, find the general solution of $\frac{dy}{dt} = \left(\frac{y}{t}\right)^2 - 2$ in explicit form. Assume that $t > 0$. The formula $\frac{1}{x^2 - x - 2} = \frac{1}{3}\left(\frac{1}{x-2} - \frac{1}{x+1}\right)$ may come in handy.

SOLUTION:

With $v = y/t, y = vt$ and $y' = v + tv'$. Substituting into the original differential equation we have

$$v + tv' = v^2 - 2$$

$$tv' = v^2 - v - 2 = (v-2)(v+1)$$

$$\frac{dv}{(v-2)(v+1)} = \frac{dt}{t}$$

$$\frac{1}{3}\int\left(\frac{dv}{v-2} - \frac{dv}{v+1}\right) = \int \frac{dt}{t}$$

$$\ln|v-2| - \ln|v+1| = 3\ln|t| + k$$

$$\ln\left|\frac{v-2}{v+1}\right| = 3\ln|t| + k = \ln t^3 + k \qquad (t > 0)$$

$$\left|\frac{v-2}{v+1}\right| = e^k t^3$$

$$\frac{y/t - 2}{y/t + 1} = Ct^3$$

$$\frac{y - 2t}{y + t} = Ct^3$$

$$y - 2t = Ct^3(y + t) = Cyt^3 + Ct^4$$

$$y(1 - Ct^3) = 2t + Ct^4$$

$$y = \frac{t(Ct^3 + 2)}{1 - Ct^3}$$

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