- This exam is worth 100 points and has 7 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5”× 11” crib sheet with writing on one side.

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0. At the top of the page containing your solution to problem 1, write the following statement and sign your name to it: “I will abide by the CU Boulder Honor Code on this exam.” **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/091725 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

   (a) $x^2 - 2xy - y^2 = 4$ is the general solution of $(x + y)y' = x - y$.

   (b) The differential equation $\frac{\mathrm{d}z}{\mathrm{d}t} - z^3 + 2z = 2$ has three equilibrium solutions that are rational numbers.

   (c) $(y'')^3 + 7y' - y = e^t$ is a second order nonlinear differential equation.

   (d) For all solutions, $y(t)$, of the differential equation $y' + y^2 = -1$, $\lim_{t\to\infty} y(t) = -1$.

   (e) If $x_1(t)$ and $x_2(t)$ are solutions of $\frac{1}{t-1}x'' + (t + 1)x' + x = 0$, then $7x_1 - 11x_2$ is also a solution.

2. [2360/091725 (14 pts)] Consider the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = x^6 - x^4$.

   (a) (3 pts) For each of the following pairs giving various classifications of differential equations, in your solutions write the correct one that the equation satisfies: linear/nonlinear; separable/nonseparable; autonomous/nonautonomous

   (b) (5 pts) For the initial condition $x(1) = 2$ and step size $1/16$, find the approximation to the true solution after one step of Euler’s method. Would your answer change if the initial condition were $x(t_0) = 2$ for any $t_0 \neq 1$? Briefly explain.

   (c) (6 pts) Find all equilibrium solutions and determine their stability.

3. [2360/091725 (17 pts)] Consider the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = \frac{1}{x},\ x > 0$.

   (a) (3 pts) Find the isocline along which the slope of the solution is 1.

   (b) (10 pts) Use the integrating factor method to find the general solution. No points for using any other method.

   (c) (4 pts) Find the solution passing through the point $(1, 1)$.

4. [2360/091725 (18 pts)] Using Newton’s second law, the velocity, u, of an object with mass m experiencing a gravitational force of mg and air resistance proportional to the velocity is given by the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = g - \frac{k}{m}u$$

   where $k > 0$ is a constant and m and g are also positive constants.

   (a) (5 pts) For any initial condition $u(t_0) = u_0$, what conclusions, if any, can be drawn from Picard’s Theorem? Justify your answer.

   (b) (10 pts) Use variation of parameters (Euler-Lagrange Two Step Method) to find the general solution to the equation. No points for using any other method.

   (c) (3 pts) Does the solution possess a steady state portion? If so, what is it? If not, explain why not.

MORE PROBLEMS BELOW/ON REVERSE

5. [2360/091725 (8 pts)] A 1000-liter holding tank is initially half full of a well-mixed solution containing 30 kilograms (kg) of Magic Potion X. When time starts ($t = 0$), the solution flows out of the tank at a rate of 5 liters (L) per hour (hr). Simultaneously, the flow into the tank is 10 L/hr. A wizard sitting above the tank is putting $(2 + \cos t)$ kg/L of Magic Potion X into the liquid flowing into the tank. Letting $x(t)$ be the amount of Magic Potion X in the tank at time t, write down, but **do not solve**, the initial value problem (IVP) whose solution is $x(t)$.

6. [2360/091725 (18 pts)] Consider the system of differential equations

$$x' = x - xy$$
$$y' = y + xy$$

(a) (4 pts) Find the h nullclines, if any exist.

(b) (4 pts) Find the v nullclines, if any exist.

(c) (4 pts) Find the equilibrium solutions, if any exist.

(d) (3 pts) Draw a phase plane in your bluebook with an arrow indicating the direction of the trajectory at the point $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

(e) (3 pts) If $x(0) = -1$ and $y(0) = 1$, find $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$.

7. [2360/091725 (15 pts)] In the homework there were several examples of using substitutions (changes of variables) to solve differential equations. Using the substitution $v = y/t$, find the general solution of $\frac{\mathrm{d}y}{\mathrm{d}t} = \left(\frac{y}{t}\right)^2 - 2$ in explicit form. Assume that $t > 0$. The formula $\frac{1}{x^2 - x - 2} = \frac{1}{3}\left(\frac{1}{x-2} - \frac{1}{x+1}\right)$ may come in handy.