1. [2350/091725 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

   (a) If $\mathbf{r}(t) = \left\langle e^{-t}\cos t, e^{-2t}\sin 2t, \dfrac{2}{t}+1 \right\rangle$, then $\lim\limits_{t\to\infty} \mathbf{r}(t) = \mathbf{k}$.

   (b) The vector $3\,\mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$ is parallel to the plane $6x - 2y + 4z = 1$.

   (c) For any vectors $\mathbf{u}, \mathbf{v}$, $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$.

   (d) If $\mathbf{v}$ is a nonzero vector, then $\mathbf{r}(t) = \langle 1,2,3\rangle + t\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is an arc length parameterization.

   (e) If $\mathbf{T}(t)$ is the unit tangent vector of a smooth curve, then the curvature is $\kappa = \|\mathrm{d}\mathbf{T}/\mathrm{d}t\|$.

   SOLUTION:

   (a) **TRUE**

$$\lim_{t\to\infty} \mathbf{r}(t) = \left\langle \lim_{t\to\infty} e^{-t}\cos t, \lim_{t\to\infty} e^{-2t}\sin 2t, \lim_{t\to\infty}\left(\frac{2}{t}+1\right)\right\rangle = \langle 0,0,1\rangle = \mathbf{k}$$

   (b) **FALSE** The normal vector to the plane is $6\,\mathbf{i} - 2\,\mathbf{j} + 4\,\mathbf{k}$ which is a scalar multiple of the given vector implying that the given vector is orthogonal to the plane, not parallel.

   (c) **TRUE**

$$|\mathbf{u}\cdot\mathbf{v}| = \|\mathbf{u}\|\|\mathbf{v}\||\cos\theta| \le \|\mathbf{u}\|\|\mathbf{v}\| \text{ since } |\cos\theta| \le 1$$

   (d) **TRUE**

$$\|\mathbf{r}'(t)\| = \left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = 1$$

   (e) **FALSE** κ is the magnitude of the rate of change of the unit tangent vector $\mathbf{T}$ with respect to arc length s, not with respect to t.

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2. [2350/091725 (14 pts)] Buzz Lightyear is positioned at $(0,1,0)$ and is attempting to return to his Star Command Base. Unfortunately, the Emperor Zurg has placed an energy shield in the shape of $x^2 + 2y^2 + z^2 = 10$ in the region surrounding Buzz to try and trap him. However, he received word from Star Command that he can safely escape the energy shield at two intersection points if he travels along the path $\mathbf{r}(t) = t\,\mathbf{i} + (1+t)\,\mathbf{j} - t\,\mathbf{k}$ where t is a real number. Find the closest such intersection point to Buzz's present position.

   SOLUTION:

   Buzz's path is a line and we seek points where the line intersects the ellipsoid. This will occur when the coordinates of the point on the line satisfy the equation of the ellipsoid. To that end,

$$t^2 + 2(1+t)^2 + (-t)^2 = 10$$

$$t^2 + 2(1+2t+t^2) + t^2 = 10$$

$$4t^2 + 4t + 2 - 10 = 4t^2 + 4t - 8 = 0$$

$$t^2 + t - 2 = (t+2)(t-1) = 0 \implies t = -2, 1$$

   The intersection points are $\mathbf{r}(-2) = -2\,\mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$ and $\mathbf{r}(1) = \mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$. The distances of these points from Buzz's initial position are, respectively,

$$\sqrt{(-2-0)^2 + (-1-1)^2 + (2-0)^2} = 2\sqrt{3} \quad \text{and} \sqrt{(1-0)^2 + (2-1)^2 + (-1-0)^2} = \sqrt{3}$$

   So the closest point to safely pass through the energy shield is $(1,2,-1)$.

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3. [2350/091725 (19 pts)] A hummingbird is flying along the path $\mathbf{r}(t) = \left\langle \frac{\sqrt{2}}{2}t^2, t+1, -\frac{1}{3}t^3 \right\rangle$, where $t \geq 0$.

(a) (3 pts) Find the hummingbird's velocity vector.

(b) (3 pts) Verify that the hummingbird's speed is given by $t^2 + 1$ and then find how far the hummingbird flies during the interval $0 \leq t \leq 3$.

(c) (3 pts) Find $\mathbf{T}(t)$ for the hummingbird.

(d) (3 pts) Is the hummingbird's acceleration vector always parallel to the xz-plane? Justify your answer.

(e) (7 pts) Given that $\mathbf{N}(t) = \frac{1}{t^2+1}\langle 1-t^2, -\sqrt{2}t, -\sqrt{2}t \rangle$, which you need not verify, find the equation of the hummingbird's osculating plane when $t = \sqrt{2}$, writing your answer in the form $ax + by + cz = d$.

SOLUTION:

(a)

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle \sqrt{2}t, 1, -t^2 \right\rangle$$

(b)

$$\|\mathbf{r}'(t)\| = \sqrt{\left(\sqrt{2}t\right)^2 + 1^2 + (-t^2)^2} = \sqrt{2t^2 + 1 + t^4} = \sqrt{(t^2+1)^2} = t^2 + 1$$

$$\text{distance flown} = \int_0^3 \left(t^2 + 1\right) \mathrm{d}t = \left(\frac{t^3}{3} + t\right)\Big|_0^3 = 12$$

(c)

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{t^2+1}\left\langle \sqrt{2}t, 1, -t^2 \right\rangle$$

(d) Yes. The hummingbird's acceleration vector is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \langle \sqrt{2}, 0, -2t \rangle$. A vector normal to the xz-plane is $\mathbf{j}$ and $\mathbf{a}(t) \cdot \mathbf{j} = 0$, implying that the acceleration vector is always parallel to the xz-plane. Alternatively, note also that there is no $\mathbf{j}$-component in $\mathbf{a}(t)$, also showing that the acceleration is parallel to the xz-plane.

(e) We need a point in the osculating plane along with its normal, which is the binormal, $\mathbf{B}(t)$. At $t = \sqrt{2}$ we have

$$\mathbf{r}\left(\sqrt{2}\right) = \left\langle \sqrt{2}, \sqrt{2}+1, -\frac{2\sqrt{2}}{3} \right\rangle$$

$$\mathbf{T}\left(\sqrt{2}\right) = \frac{1}{3}\langle 2, 1, -2 \rangle$$

$$\mathbf{N}\left(\sqrt{2}\right) = \frac{1}{3}\langle -1, -2, -2 \rangle$$

$$\mathbf{B}\left(\sqrt{2}\right) = \mathbf{T}\left(\sqrt{2}\right) \times \mathbf{N}\left(\sqrt{2}\right) = \frac{1}{9}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ -1 & -2 & -2 \end{vmatrix} = \frac{1}{9}\langle -6, 6, -3 \rangle = \frac{1}{3}\langle -2, 2, -1 \rangle$$

Using this information, the equation of the plane is

$$-\frac{2}{3}\left(x - \sqrt{2}\right) + \frac{2}{3}\left(y - \sqrt{2} - 1\right) - \frac{1}{3}\left(z + \frac{2}{3}\sqrt{2}\right) = 0$$

$$-2x + 2y - z = \frac{2}{3}\sqrt{2} + 2$$

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4. [2350/091725 (14 pts)] Find the equation of the plane (write your answer in the form $ax + by + cz = d$) containing the intersecting lines

$$L_1 : x = 2 - s, y = s, z = 2 \quad \text{and} \quad L_2 : x - 1 = 1 - y = \frac{z}{2}$$

SOLUTION:

We need a point in the plane and thus on both lines and a normal to the plane. The direction vector of L_1 is $\mathbf{v}_1 = \langle -1, 1, 0 \rangle$. The parametric form of L_2 is $x = 1 + t, y = 1 - t, z = 2t$ having the direction vector $\mathbf{v}_2 = \langle 1, -1, 2 \rangle$. A vector normal to both of these and therefore normal to the plane we seek is $\mathbf{v}_1 \times \mathbf{v}_2 = \langle 2, 2, 0 \rangle$. To find the point of intersection of the lines we solve the system

$$\begin{aligned} 1 + t &= 2 - s \\ 1 - t &= s \\ 2t &= 2 \end{aligned}$$

The last equation requires $t = 1$, forcing $s = 0$ in the second equation. These values satisfy the first equation so the point of intersection is $(2, 0, 2)$. The equation of the plane is then

$$2(x-2) + 2(y-0) + 0(z-2) = 0 \implies 2x - 4 + 2y = 0 \implies x + y = 2$$

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5. [2350/091725 (16 pts)] You are trapped in a box with a hinged lid on top. Because you are in such a tight space, you can only push up on the lid at a 45 degree angle using a force, $\mathbf{F}$, of magnitude 2.

   (a) (4 pts) If the magnitude of the torque, $\boldsymbol{\tau}$, required to rotate the lid around the hinge, opening the box, is $\sqrt{6}$, how far from the hinge should you apply the force?

   (b) (4 pts) If the same force applied at the same angle moves an object the same distance as that found in part (a), would the work be the same as the magnitude of the torque? Justify your answer.

   (c) (4 pts) Find the area of the parallelogram formed by the force vector and the vector depicting where you applied the force in part (a).

   (d) (4 pts) Is the work found in part (b) equal to $\text{proj}_{\mathbf{D}}\mathbf{F}$ where $\mathbf{D}$ is the vector giving the displacement of the object? Justify your answer.

SOLUTION:

(a)

$$\|\boldsymbol{\tau}\| = \|\mathbf{F}\|\|\mathbf{r}\| \sin\theta \implies \|\mathbf{r}\| = \frac{\|\boldsymbol{\tau}\|}{\|\mathbf{F}\| \sin\theta} = \frac{\sqrt{6}}{2\,(\sin 45^\circ)} = \frac{\sqrt{6}}{2\left(\sqrt{2}/2\right)} = \sqrt{3}$$

(b) Yes.

$$W = \|\mathbf{F}\|\|\mathbf{D}\| \cos\theta = 2\sqrt{3}\cos 45^\circ = 2\sqrt{3}\left(\sqrt{2}/2\right) = \sqrt{6}$$

(c) Area $= \|\mathbf{r} \times \mathbf{F}\| = \|\boldsymbol{\tau}\| = \sqrt{6}$

(d) No. Work is a scalar and $\text{proj}_{\mathbf{D}}\mathbf{F}$ is a vector.

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6. [2350/091725 (15 pts)] A particle is traveling along a path, $\mathbf{r}(t)$, experiencing an acceleration of $\mathbf{a}(t) = 2t\,\mathbf{i} + e^t\,\mathbf{j}$. Assume that $t \geq 0$ and $\mathbf{v}(0) = \mathbf{i} + \mathbf{j}$.

   (a) Find the time(s), if any, when the speed of the particle is not changing.

   (b) Is the direction of the particle always changing? Justify your answer.

SOLUTION:

We need to find the tangential and normal components of the acceleration. To do this we need the acceleration vector [$\mathbf{r}''(t)$, given] and the velocity [$\mathbf{r}'(t)$], found as

$$\mathbf{v}(t) = \int \mathbf{v}'(t)\,\mathrm{d}t = \int \mathbf{a}(t)\,\mathrm{d}t = \int \left(2t\,\mathbf{i} + e^t\,\mathbf{j}\right)\mathrm{d}t$$
$$= \left(\int 2t\,\mathrm{d}t\right)\mathbf{i} + \left(\int e^t\,\mathrm{d}t\right)\mathbf{j}$$
$$= \left(t^2 + c_1\right)\mathbf{i} + \left(e^t + c_2\right)\mathbf{j}$$
$$\mathbf{v}(0) = c_1\,\mathbf{i} + (1 + c_2)\,\mathbf{j} = \mathbf{i} + \mathbf{j} \implies c_1 = 1, c_2 = 0$$
$$\mathbf{r}'(t) = \mathbf{v}(t) = \left(t^2 + 1\right)\mathbf{i} + e^t\,\mathbf{j}$$

(a)

$$a_T = \frac{\mathbf{r}' \cdot \mathbf{r}''}{\|\mathbf{r}'\|} = \frac{\langle t^2 + 1, e^t\rangle \cdot \langle 2t, e^t\rangle}{\sqrt{\left(t^2+1\right)^2 + (e^t)^2}} = \frac{2t^3 + 2t + e^{2t}}{\sqrt{t^4 + 2t^2 + 1 + e^{2t}}} > 0 \quad \text{for all } t \geq 0$$

There are no points where the speed is not changing since $a_T > 0$ for all $t \geq 0$. An alternative approach, resulting in the same equation, is

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}(t)\| = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\left(t^2+1\right)^2 + (e^t)^2} = \frac{2t^3 + 2t + e^{2t}}{\sqrt{t^4 + 2t^2 + 1 + e^{2t}}}$$

(b)

$$a_N = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|} = \frac{\left\|\langle t^2 + 1, e^t\rangle \times \langle 2t, e^t\rangle\right\|}{\sqrt{\left(t^2+1\right)^2 + (e^t)^2}}$$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^2+1 & e^t & 0 \\ 2t & e^t & 0 \end{vmatrix} = \left(t^2e^t + e^t - 2te^t\right)\mathbf{k} = e^t\left(t^2 - 2t + 1\right)\mathbf{k} = e^t\left(t-1\right)^2\mathbf{k} \implies \|\mathbf{r}' \times \mathbf{r}''\| = e^t(t-1)^2$$

$$a_N = \frac{e^t(t-1)^2}{\sqrt{t^4 + 2t^2 + 1 + e^{2t}}}$$

No. When $t = 1, a_N = 0$ implying that the direction of the particle is not changing at that time.

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7. [2350/091725 (12 pts)] Consider the equation $ax^2 + by^2 + cz^2 + 2z = 2$. For each part, pick the correct surface from the list below that is described when the constants take on the given values. Write your answers in a single column separate from any work you do to arrive at your answers. No partial credit will be awarded.

cylinder   hyperboloid of one sheet   paraboloid   ellipsoid   plane   cone   hyperboloid of two sheets

(a) $a > 0, b > 0, c = -1/2$

(b) $a \neq 0, b = c = 0$

(c) $a \neq 0, b \neq 0, c = 0$

(d) $a = b = c = 0$

(e) $a = 1, b = c = -1$

(f) $a = c = 1, b = -1$

SOLUTION:

(a) $ax^2 + by^2 - \frac{1}{2}z^2 + 2z = 2 \implies ax^2 + by^2 - \frac{(z-2)^2}{2} = 0$; cone

(b) $ax^2 + 2z = 2$; cylinder

(c) $ax^2 + by^2 + 2z = 2$; paraboloid

(d) $2z = 2 \implies z = 1$; plane

(e) $x^2 - y^2 - z^2 + 2z = 2 \implies x^2 - y^2 - (z-1)^2 = 1 \implies -x^2 + y^2 + (z-1)^2 = -1$; hyperboloid of two sheets

(f) $x^2 - y^2 + z^2 + 2z = 2 \implies x^2 - y^2 + (z+1)^2 = 3$; hyperboloid of one sheet

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