1. (25 pts) Parts (a) - (d) are not related to each other.

   (a) i. Fully simplify the rational expression $\dfrac{x^2+2x-3}{x^2-5x+4}$.

   ii. Identify all value(s) of x, if any, for which the original expression is undefined.

   iii. Identify all value(s) of x, if any, for which the simplified expression is undefined.

Solution:

i.

$$\frac{x^2+2x-3}{x^2-5x+4} = \frac{(x-1)(x+3)}{(x-1)(x-4)} = \boxed{\frac{x+3}{x-4}}$$

ii. The original expression is undefined due to division by zero for both $x = 1$ and $x = 4$ according to the factorization of that expression's denominator.

iii. The simplified expression is undefined due to division by zero for $x = 4$ only.

   (b) Find all solutions, if any, of the equation $x^{9/4} + 6x^{1/4} = 5x^{5/4}$.

Solution:

$$x^{9/4} + 6x^{1/4} = 5x^{5/4}$$

$$x^{9/4} - 5x^{5/4} + 6x^{1/4} = 0$$

$$x^{1/4}(x^2 - 5x + 6) = 0$$

$$x^{1/4}(x-2)(x-3) = 0$$

Any value of x for which any of the preceding three factors equals zero is a solution to the original equation.

$$x^{1/4} = 0 \;\Rightarrow\; x = 0$$

$$x - 2 = 0 \;\Rightarrow\; x = 2$$

$$x - 3 = 0 \;\Rightarrow\; x = 3$$

Therefore, the set of solutions is $\boxed{x = 0, 2, 3}$

(c) Fully simplify the following complex fraction by expressing it as a rational expression (a quotient of two polynomials):

$$\frac{\frac{1}{x+1}+\frac{2}{x+2}}{\frac{3}{x+3}}$$

You may express the polynomials in either factored form or expanded form.

Solution:

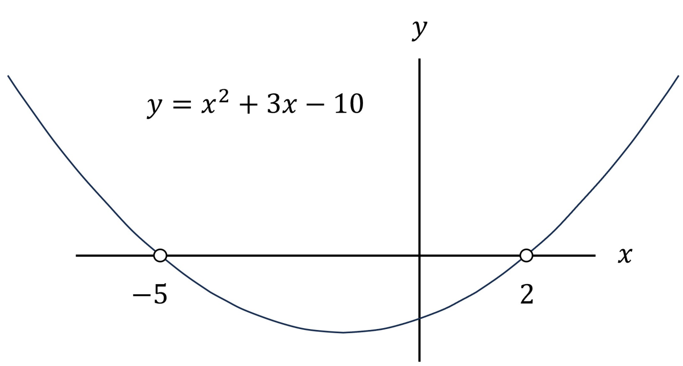
$$\begin{aligned}\frac{\frac{1}{x+1}+\frac{2}{x+2}}{\frac{3}{x+3}} &= \left(\frac{1}{x+1}+\frac{2}{x+2}\right)\left(\frac{x+3}{3}\right)\\ &= \frac{(x+2)+2(x+1)}{(x+1)(x+2)}\cdot\frac{x+3}{3}\\ &= \boxed{\frac{(3x+4)(x+3)}{3(x+1)(x+2)} = \frac{3x^2+13x+12}{3x^2+9x+6}}\end{aligned}$$

(d) Solve the inequality $x^2 > 10 - 3x$. Express your answer using interval notation.

Solution:

$$\begin{aligned}x^2 &> 10 - 3x\\ x^2+3x-10 &> 0\\ (x-2)(x+5) &> 0\end{aligned}$$

Solution Method 1: Graph



The preceding graph indicates that $x^2 + 3x - 10 > 0$ on $\boxed{(-\infty,-5)\cup(2,\infty)}$

Solution Method 2: Table

$$x^2 + 3x - 10 = (x-2)(x+5) > 0$$

	$x < -5$	$-5 < x < 2$	$x > 2$
$x - 2$	$-$	$-$	$+$
$x + 5$	$-$	$+$	$+$
$(x-2)(x+5)$	$+$	$-$	$+$

x ; $x = -5$; $x = 2$

The preceding table shows the signs of the terms $(x-2)$, $(x-5)$, and $(x-2)(x-5) = x^2 + 3x - 10$ on each of the following three subintervals of $(-\infty, \infty)$: $(-\infty, -5)$, $(-5, 2)$, and $(2, \infty)$.

The table indicates that $(x-2)(x+5) = x^2 + 3x - 10 > 0$ on $\boxed{(-\infty, -5) \cup (2, \infty)}$

2. (25 pts) Parts (a) and (b) are not related to each other.

   (a) For parts i-iii, let point A be $(-3,-1)$, let point B be $(2,2)$, let segment AB be the line segment connecting points A and B, and let point M be the midpoint of segment AB.

      i. Find the (x, y) coordinates of point M.

      Solution:

      The general expression for the midpoint of the line segment connecting the points (x_1, y_1) and (x_2, y_2) is

      $$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

      Substituting the coordinate values of points A and B leads to the following midpoint of segment AB:

      $$\left(\frac{-3+2}{2}, \frac{-1+2}{2}\right) = \boxed{\left(-\frac{1}{2}, \frac{1}{2}\right)}$$

      ii. Find the length of segment AB.

      Solution:

      The length of segment AB is the distance between points A and B, so we'll apply the distance formula.

      $$\begin{aligned} D &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(2-(-3))^2 + (2-(-1))^2} \\ &= \sqrt{5^2 + 3^2} \\ &= \boxed{\sqrt{34}} \end{aligned}$$

iii. Find an equation of the line that is perpendicular to segment AB and passes through point A.

Solution:

The first step is to determine the slope of segment AB. We'll let m_1 denote that slope.

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - (-1)}{2 - (-3)} = \frac{3}{5}$$

The slopes of two perpendicular lines are negative reciprocals of each other. Since the slope of segment AB is $3/5$, the slope of a line that is perpendicular to segment AB is $-5/3$.

The point-slope form of the equation of the line passing through a point (x_0, y_0) is $y - y_0 = m(x - x_0)$.

Since point A lies on the line that is perpendicular to segment AB at point A, we'll use $(x_0, y_0) = (-3, -1)$, which are the coordinates of point A.

Therefore, an equation for the line that is perpendicular to segment AB and passes through point A is

$y - (-1) = -\frac{5}{3}(x - (-3))$, which is $\boxed{y + 1 = -\frac{5}{3}(x + 3)}$

(b) Find the center and radius of the circle whose equation is $x^2 - 8x + y^2 = 20$.

Hint: Complete the square.

Solution:

The equation of a circle centered at a point (h, k) with a radius of r is $(x-h)^2 + (y-k)^2 = r^2$.

First, we'll complete the square for $x^2 - 8x$. The coefficient to x is -8, so we'll divide that number by two, then square the result:

$$\left(\frac{-8}{2}\right)^2 = (-4)^2 = 16$$

So, to complete the square, we add 16 to each side of the original equation, as follows:

$$x^2 - 8x + y^2 = 20$$
$$x^2 - 8x + 16 + y^2 = 20 + 16 = 36$$

Since the only term related to the variable y is y^2, which is already a squared term, there is no need to complete the square for the variable y.

Therefore, the original equation for the given circle can be expressed as follows, which corresponds to the general form of an equation of a circle:

$$(x-4)^2 + y^2 = 6^2$$

Comparing this result to the general form of an equation of a circle, we see that $h = 4$, $k = 0$, and $r = 6$. Therefore, the center and radius of the given circle are as follows:

Center: $\boxed{(4, 0)}$

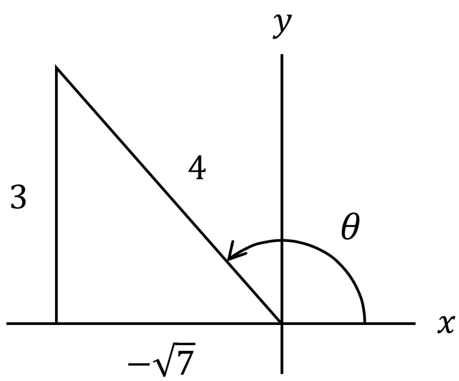
Radius: $\boxed{6}$

3. (20 pts) Parts (a) and (b) are not related to each other.

(a) If $\sin\theta = 3/4$ and θ is on the interval $(\pi/2, \pi)$, find the value of $\cot\theta$.

Solution:

Since θ is on the interval $(\pi/2, \pi)$, there is a triangle in quadrant II that is associated with angle θ, as shown in the following figure.



Since the triangle is in quadrant II, the *adjacent* (horizontal) component of the triangle is negative and the *opposite* (vertical) component of the triangle is positive.

Since $\sin\theta = 3/4$, the ratio of the *opposite* component to the *hypotentuse* component is $3/4$. Therefore, we can let the *opposite* component equal 3 and the *hypotenuse* component equal 4, as depicted in the preceding figure.

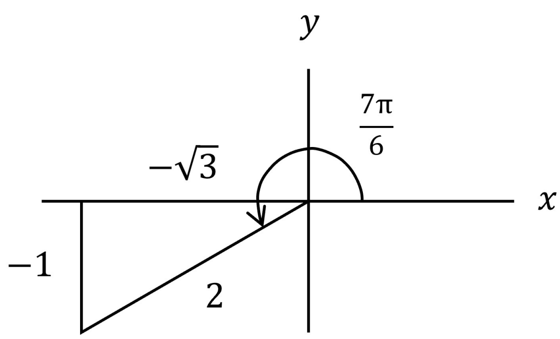
The Pythagorean Theorem indicates that if the length of the hypotenuse of a right triangle is 4 and the length of one leg of that right triangle is 3, then the length of the other leg is $\sqrt{4^2 - 3^2} = \sqrt{16 - 9} = \sqrt{7}$. Since the *adjacent* component of the triangle in the preceding figure is negative, the value of the *adjacent* component is $-\sqrt{7}$, as shown in the figure.

Therefore, $\cot\theta = \dfrac{\textit{adjacent}}{\textit{opposite}} = \dfrac{-\sqrt{7}}{3} = \boxed{-\dfrac{\sqrt{7}}{3}}$

(b) Evaluate $\csc\left(\frac{7\pi}{6}\right)$

Solution:

Since $\pi < \frac{7\pi}{6} < \frac{3\pi}{2}$, the angle $\frac{7\pi}{6}$ lies in Quadrant III, as drawn in the following figure.



The reference angle is $\frac{7\pi}{6} - \pi = \frac{\pi}{6}$, which is the $30°$ angle in the special $30° - 60° - 90°$ right triangle. The dimensions of such a triangle are proportional to 1, $\sqrt{3}$, and 2, which leads to the set of dimensions displayed in the figure, with the negative values of the *adjacent* and *opposite* components of the triangle accounting for the fact that the triangle is in Quadrant III.

It follows from the figure that $\csc\left(\frac{7\pi}{6}\right) = \frac{\textit{hypotenuse}}{\textit{opposite}} = \frac{2}{-1} = \boxed{-2}$

4. (16 pts) Parts (a) and (b) are not related to each other.

(a) Use the trigonometric identity for $\cos(\alpha + \beta)$ and the fact that $75° = 30° + 45°$ to find the exact value of $\cos(75°)$.

Solution:

$$\cos(75°) = \cos(30° + 45°) = \cos(30°)\cos(45°) - \sin(30°)\sin(45°)$$
$$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}}$$
$$= \boxed{\frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}}$$

(b) On a circle of radius r, the length of the arc subtended by an angle of $15°$ is $7\pi/4$ inches. Find the value of r, including the correct unit of measurement.

Solution:

Accoring to the formula sheet on the final page of the exam, $L = \theta r$, where θ is expressed in radians.

$$\theta = (15°)\left(\frac{\pi \text{ radians}}{180°}\right) = \frac{\pi}{12} \text{ radians}$$

So, in the equation $L = \theta r$, we have $\theta = \pi/12$ radians and $L = 7\pi/4$ inches. Therefore,

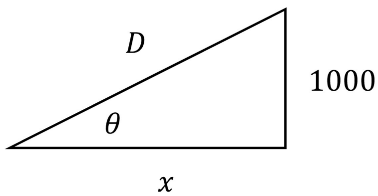
$$\frac{7\pi}{4} = \frac{\pi}{12} \cdot r$$
$$r = \frac{7\pi}{4} \cdot \frac{12}{\pi} = \boxed{21 \text{ inches}}$$

5. (14 pts) Suppose that you are standing on a flat horizontal path that leads to the base of a mountain, and that the top of the mountain is 1000 feet directly above the mountain's base. Use the following variable assignments:

- Let x represent the horizontal distance, in feet, between you and the base of the mountain
- Let D represent the straight-line (diagonal) distance, in feet, between you and the top of the mountain
- Let θ represent the angle, in radians, between the path and your line of sight to the top of the mountain

(a) Draw a right triangle to depict the situation, including correct labels for the angle θ and the corresponding side lengths x, D, and 1000.

Solution:



(b) Find an expression for x in terms of D. Your expression should not include the variable θ.

Solution:

The Pythagorean Theorem can be applied to the preceding right triangle as follows:

$$x^2 + 1000^2 = D^2$$
$$x^2 = D^2 - 1000^2$$

Therefore, $\boxed{x = \sqrt{D^2 - 1000^2}}$

(In the final step, only the positive root makes physical sense in this problem.)

(c) Find an expression for D in terms of θ. Your expression should include a trigonometric function and it should not include the variable x.

Solution:

According to the figure above, $\sin\theta = \dfrac{\textit{opposite}}{\textit{hypotenuse}} = \dfrac{1000}{D}$.

Therefore, $\boxed{D = \dfrac{1000}{\sin\theta} = 1000\csc\theta}$