1. (30 points) Evaluate the following limits and simplify your answers. If a limit does not exist, clearly state this. If you use a theorem, clearly state its name and show that its hypotheses are satisfied.
(Reminder: You may not use L'Hopital's Rule in any solutions on this exam.)

   (a) $\displaystyle\lim_{x\to 1} \frac{3-\sqrt{6+3x}}{x-1}$

   (b) $\displaystyle\lim_{x\to 1} \frac{(x-2)|x-1|}{x^2-3x+2}$

   (c) $\displaystyle\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right)$.

Solution:

(a) We will evaluate this limit by first multiplying the numerator and denominator by the conjugate of the numerator, in an effort to eliminate the $0/0$-indeterminate form:

$$\begin{aligned}
\lim_{x\to 1} \frac{3-\sqrt{6+3x}}{x-1} &= \lim_{x\to 1} \frac{3-\sqrt{6+3x}}{x-1} \cdot \frac{3+\sqrt{6+3x}}{3+\sqrt{6+3x}} \\
&= \lim_{x\to 1} \frac{9-(6+3x)}{(x-1)(3+\sqrt{6+3x})} \\
&= \lim_{x\to 1} \frac{3(1-x)}{(x-1)(3+\sqrt{6+3x})} \\
&= \lim_{x\to 1} \frac{3(1-x)}{(x-1)(3+\sqrt{6+3x})} \\
&= \lim_{x\to 1} \frac{-3}{3+\sqrt{6+3x}} \\
&= \frac{-3}{3+\sqrt{6+3}} \\
&= -\frac{1}{2}.
\end{aligned}$$

(b) We see that

$$\frac{(x-2)|x-1|}{x^2-3x+2} = \frac{(x-2)|x-1|}{(x-1)(x-2)} = \frac{|x-1|}{x-1}, \text{ if } x \neq 1, 2.$$

We now check the corresponding one-sided limits:

$$\begin{aligned}
\lim_{x\to 1^+} \frac{(x-2)|x-1|}{x^2-3x+2} &= \lim_{x\to 1^+} \frac{|x-1|}{x-1} \\
&= \lim_{x\to 1^+} \frac{x-1}{x-1} \\
&= \lim_{x\to 1^+} 1 \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to 1^-}\frac{(x-2)|x-1|}{x^2-3x+2} &= \lim_{x\to 1^-}\frac{|x-1|}{x-1}\\ &= \lim_{x\to 1^-}\frac{-(x-1)}{x-1}\\ &= \lim_{x\to 1^-} -1\\ &= -1.\end{aligned}$$

Since the one-sided limits disagree, then $\lim_{x\to 1}\dfrac{(x-2)|x-1|}{x^2-3x+2}$ does not exist.

(c) We will apply the Squeeze Theorem to $\lim_{x\to 0} x^2\cos\left(\dfrac{1}{x^2}\right)$. Note that

$$-x^2 \le x^2\cos\left(\frac{1}{x^2}\right) \le x^2$$

for all $x \neq 0$, and that

$$\lim_{x\to 0} -x^2 = 0 = \lim_{x\to 0} x^2.$$

So, we know $\lim_{x\to 0} x^2\cos\left(\dfrac{1}{x^2}\right) = 0$.

2. (22 points) Answer each of the following problems. (a) and (b) are unrelated to one another.

(a) Let $f(\theta) = 6\cos(2\theta) - 1$.

i. What is the range of f? Write your final answer in interval notation.

ii. Solve the inequality $f(\theta) > 2$ for θ in the interval $[0, \pi]$. Write your final answer in interval notation.

(b) Suppose $\cos t = \dfrac{3}{7}$ and $\sin t < 0$. Find $\sin(2t)$.

Solution:

(a) i. The range of $\cos(2\theta)$ is $[-1, 1]$. If we multiply by 6 to get $6\cos(2\theta)$, the amplitude is now 6 and the range is $[-6, 6]$. Subtracting 1 shifts the function down one unit, so the range of $f(\theta)$ is $[-7, 5]$.

ii. We wish to solve

$$6\cos(2\theta) - 1 > 2,$$

which simplifies to

$$\cos(2\theta) > \frac{1}{2}.$$

We know that $\cos(2\theta) = \frac{1}{2}$ when $2\theta = \pi/3, 5\pi/3$ when restricting our attention to values in $[0, 2\pi]$. Since we are interested in $\cos(2\theta) > \frac{1}{2}$, then we have

$$0 \le 2\theta < \frac{\pi}{3} \text{ or } \frac{5\pi}{3} < 2\theta \le 2\pi.$$

Solving for θ gives

$$0 \le \theta < \frac{\pi}{6} \text{ or } \frac{5\pi}{6} < \theta \le \pi,$$

or

$$\left[0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right].$$

(b) Recall that $\sin(2t) = 2\sin(t)\cos(t)$. So, we need to first find $\sin(t)$. We are given that $\sin(t) < 0$. Using a Pythagorean identity, we have

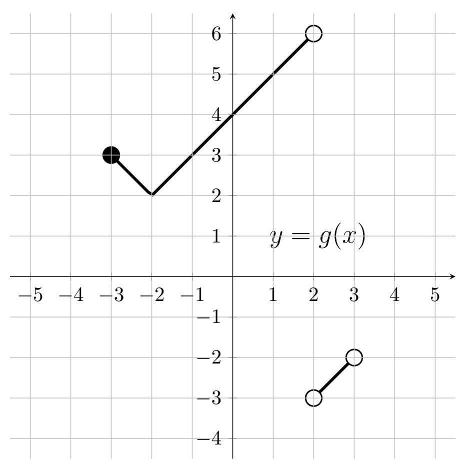
$$\begin{aligned} \sin^2 t + \cos^2 t &= 1 \\ \sin^2 t + \left(\frac{3}{7}\right)^2 &= 1 \\ \sin^2 t &= 1 - \frac{9}{49} \\ \sin^2 t &= \frac{40}{49} \\ \sin t &= -\frac{2\sqrt{10}}{7}. \end{aligned}$$

Thus,

$$\sin(2t) = 2\sin(t)\cos(t) = 2\left(-\frac{2\sqrt{10}}{7}\right)\left(\frac{3}{7}\right) = -\frac{12\sqrt{10}}{49}.$$

3. (14 points) Use the table of values of $r(x)$ and the graph of $y = g(x)$ provided here to answer each of the problems below. Assume that $r(x)$ is continuous for all real numbers. Also assume that the entire graph of $y = g(x)$ is represented in the graph below. No justifications are required for this problem.

x	0	1	2	3	4	5
$r(x)$	3	-2	4	3	1	0



(a) State the domain of $g(x)$ using interval notation.

(b) State the range of $g(x)$ using interval notation.

(c) Determine the value of $(g \circ r)(1)$.

(d) Determine the value of $(r \circ g)(1)$.

(e) Assuming $r(x)$ is an odd function, evaluate $r(-2)$.

(f) Evaluate $\lim\limits_{x \to 2^-} g(x)$, if it exists. (If it does not exist, state Does Not Exist.)

(g) Evaluate $\lim\limits_{x \to 1} (g \circ r)(x)$, if it exists. (If it does not exist, state Does Not Exist.)

Solution:

(a) $[-3, 2) \cup (2, 3)$

(b) $(-3, -2) \cup [2, 6)$

(c) $(g \circ r)(1) = g(r(1)) = g(-2) = 2$.

(d) $(r \circ g)(1) = r(g(1)) = r(5) = 0$.

(e) $r(-2) = -r(2) = -4$.

(f) $\lim_{x \to 2^-} g(x) = 6$.

(g) Since r is continuous at $x = 1$ and g is continuous at $x = r(1) = -2$, then $\lim_{x \to 1} (g \circ r)(x) = (g \circ r)(1) = 2$.

4. (20 points) Consider the following function in terms of constant a:

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & \text{if } x < 2 \\ ax^2 - 5x, & \text{if } x \geq 2 \end{cases}$$

(a) If $a = 3$, what type of discontinuity does f have at $x = 2$? (Justify your answer with the appropriate limit(s).)

(b) Determine all values of a such that $f(x)$ is continuous at $x = 2$. (Justify your answer with the definition of continuity.)

Solution:

(a) We see that

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 3x^2 - 5x = 2$$

and

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2^-} x + 2 = 4.$$

So, f has a **jump discontinuity** at $x = 2$, since the one-sided limits exist but disagree.

(b) In order for f to be continuous at $x = 2$, we need $\lim_{x \to 2} f(x) = f(2)$. We note that

- $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} ax^2 - 5x = 4a - 10$
- $f(2) = 4a - 10$, and
- $\lim_{x \to 2^-} f(x) = 4$ (from above).

So, we need all values of a such that $4a - 10 = 4$.The only solution is $a = 7/2$. That is, $a = 7/2$ is the only value of a such that f is continuous at $x = 2$.

5. (14 points) Consider $h(x) = \dfrac{x + 30}{x + 2}$ and $j(x) = x + \dfrac{1}{x}$.

(a) Is $h(x)$ even, odd, or neither? Justify your answer.

(b) State the domain of $h(x)$ in interval notation.

(c) Determine $(h \circ j)(x)$ and give its domain in interval notation. (Your expression for $(h \circ j)(x)$ should be fully simplified with no fractions inside fractions.)

Solution:

(a) $h(-x) = \frac{-x+30}{-x+2}$ is neither $h(x)$ not $-h(x)$, so h is neither even nor odd.

(b) $(-\infty, -2) \cup (-2, \infty)$

(c)

$$
\begin{aligned}
(h \circ j)(x) &= h(j(x)) \\
&= \frac{x + \frac{1}{x} + 30}{x + \frac{1}{x} + 2} \cdot \frac{x}{x} \\
&= \frac{x^2 + 30x + 1}{x^2 + 2x + 1}.
\end{aligned}
$$

Since $x^2 + 2x + 1 = (x+1)^2$, we see that -1 must be excluded from the domain. We also recall that the domain will be all x-values in the domain of j such that $j(x)$ is in the domain of h. We can scrutinize the unsimplifed form of our composition above to see that the domain is $(-\infty, -1) \cup (-1, 0) \cup (0, \infty)$.