1. (**30 pts**) Evaluate the following. You may use any calc 1 techniques.

   (a) Evaluate $\displaystyle\lim_{x\to 0}\frac{xe^x}{e^x-1}$

   (b) Evaluate $\displaystyle\lim_{x\to\frac{\pi}{2}^-}\frac{\sec x}{\tan x}$

   (c) Evaluate $\displaystyle\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

Solution:

(a) Note that this is a 0/0 indeterminate limit. So, we will use L'Hopital's rule.

$$\lim_{x\to 0}\frac{xe^x}{e^x-1} \overset{LH}{=} \lim_{x\to 0}\frac{xe^x+e^x}{e^x}$$

$$= \lim_{x\to 0}\frac{e^x(x+1)}{e^x}$$

$$= \lim_{x\to 0} x+1 = \boxed{1}$$

(b) For this limit, we start by rewriting this limit in terms of $\sin x$ and $\cos x$.

$$\lim_{x\to\frac{\pi}{2}^-}\frac{\sec x}{\tan x} = \lim_{x\to\frac{\pi}{2}^-}\frac{1}{\cos x}\cdot\frac{\cos x}{\sin x} = \lim_{x\to\frac{\pi}{2}^-}\frac{1}{\sin x} = \boxed{1}$$

(c) For this problem, we let $L = \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$ and take the log.

$$\ln L = \lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right)$$

This is an $\infty\cdot 0$, so we re-write the limit to use L'Hopital's rule:

$$\lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}}$$

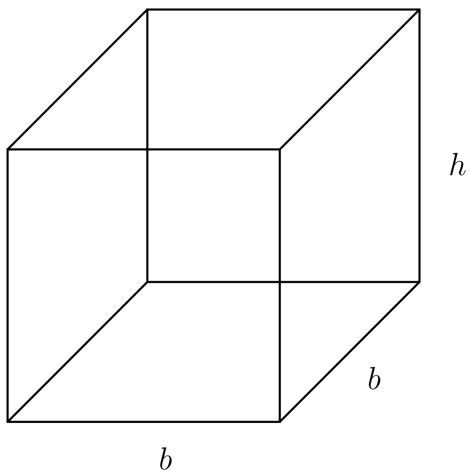
$$\overset{LH}{=} \lim_{x\to\infty}\frac{\frac{1}{1+2/x}\cdot(-2x^{-2})}{(-x^{-2})}$$

$$= \lim_{x\to\infty}\frac{2}{1+\frac{2}{x}} = 2$$

So, $\ln L = 2$, and $L = \boxed{e^2}$

2. (**15 pts**) A box with a square base and open top must have a volume of 32,000cm^3. Find the dimensions of the box that minimize the amount of material used.

Solution: Let's call the side length of the base b and the height h.



We are given a restriction on the volume, $V = 32,000 = b^2h$. We are told we want to minimize the amount of material used. Since there is no top, we want to minimize $A = b^2 + 4bh$. Using the volume equation, $h = \dfrac{32000}{b^2}$. Plugging in to our A equation, we get

$$A(b) = b^2 + \frac{4 \cdot 32000}{b}$$

$$A'(b) = 2b - 128000b^{-2}$$

To find critical points, we need to find b such that $A'(b) = 0$.

$$0 = 2b - 128,000b^{-2}$$

$$128,000b^{-2} = 2b$$

$$64,000 = b^3$$

$$b = 40\text{cm}$$

We check that this is a minimum using the second derivative test: $A''(b) = 2 + 256000b^{-3} > 0$ for $b > 0$, so $A(b)$ is concave up and we have a minimum. Plugging back in to find the height: $h = 32,000/(40^2) = 20$cm. The dimensions that minimize the amount of material used are: $\boxed{40 \times 40 \times 20\text{cm}}$.

3. (**26 pts**) For the following problems, consider the decreasing function $f(x) = \dfrac{-\sqrt{e^x}}{2}$.

(a) Find the domain and range of $f(x)$.

(b) Find $f^{-1}(x)$. Indicate the domain and range of the inverse function.

(c) Find the equation of the tangent line to $f^{-1}(x)$ when $x = \dfrac{-1}{2}$.

Solution:

(a) The domain of f is $(-\infty, \infty)$, because $e^x > 0$ for all x. The range is $(-\infty, 0)$.

(b) To find the inverse, we swap x and y and solve for y:

$$\begin{aligned} x &= \frac{-\sqrt{e^y}}{2} \\ -2x &= \sqrt{e^y} \\ 4x^2 &= e^y \\ y &= \ln\left(4x^2\right) \end{aligned}$$

So, $\boxed{f^{-1}(x) = \ln\left(4x^2\right)}$. The domain is $(-\infty, 0)$ and the range is $(-\infty, \infty)$.

(c) To find the equation of the tangent line, we need a point and the slope. To find the coordinate point, we calculate $f^{-1}\left(-\frac{1}{2}\right) = \ln\left(4 \cdot \frac{1}{4}\right) = \ln(1) = 0$. So the point is $\left(-\frac{1}{2}, 0\right)$.
To find the slope, we can either find $(f^{-1})'(-\frac{1}{2})$ or $\frac{1}{f'(0)}$.

$$\begin{aligned} (f^{-1})'(x) &= \frac{1}{4x^2} \cdot 8x \\ &= \frac{2}{x} \\ (f^{-1})'\left(-\frac{1}{2}\right) &= -4 \end{aligned}$$

So, the equation of the tangent line is $\boxed{y = -4x - 2}$.

4. (**36 pts**) The following problems are unrelated. You may use any calc 1 techniques.

(a) Compute $\displaystyle\int \frac{x}{\sqrt{16x^2+1}}\,\mathrm{d}x$

(b) Find $\dfrac{dy}{dx}$ if $y = \dfrac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}$ Hint: use logarithmic differentiation.

(c) Evaluate $\displaystyle\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}}\,\mathrm{d}x$

Solution:

(a) To evaluate this integral we do a substitution: $u = 16x^2+1$, and $\mathrm{d}u = 32x\,\mathrm{d}x$.

$$\begin{aligned}\int \frac{x}{\sqrt{u}}\frac{\mathrm{d}u}{32x} &= \frac{1}{32}\int u^{-1/2}\,\mathrm{d}x \\ &= \frac{1}{32}\left[2u^{1/2}\right]+C \\ &= \boxed{\frac{\sqrt{16x^2+1}}{16}+C}\end{aligned}$$

(b) We will use logarithmic differentiation to find $\dfrac{dy}{dx}$.

$$\begin{aligned}\ln(y) &= \ln\left(\frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}\right) \\ &= 4\ln\big(x^2+1\big) - 3\ln(2x+1) - 5\ln(3x-1)\end{aligned}$$

Now that we have simplified, we take the derivative of both sides:

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= 4\left(\frac{1}{x^2+1}\right)\cdot 2x - 3\left(\frac{1}{2x+1}\right)\cdot 2 - 5\left(\frac{1}{3x-1}\right)\cdot 3 \\ &= \left(\frac{8x}{x^2+1}\right) - \left(\frac{6}{2x+1}\right) - \left(\frac{15}{3x-1}\right) \\ \frac{dy}{dx} &= y\left[\left(\frac{8x}{x^2+1}\right) - \left(\frac{6}{2x+1}\right) - \left(\frac{15}{3x-1}\right)\right] \\ &= \boxed{\frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}\left[\left(\frac{8x}{x^2+1}\right) - \left(\frac{6}{2x+1}\right) - \left(\frac{15}{3x-1}\right)\right]}\end{aligned}$$

(c) $\displaystyle\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}}\,\mathrm{d}x = \left[4\arcsin x\right]_{1/2}^{1/\sqrt{2}} = 4\left[\frac{\pi}{4}-\frac{\pi}{6}\right] = \frac{4\pi}{12} = \boxed{\frac{\pi}{3}}$

5. (**18 pts**) Consider a bacteria culture that grows at a rate proportional to its size. The size of the culture increases by 50% in one hour. Let $P(t)$ be the number of bacteria after t hours.

(a) Find the relative growth rate of the culture (k).

(b) Find an expression for $P(t)$. Fully simplify your answer.

(c) If the number of cells in the culture is 900 after two hours, what was the initial population of the culture?

Solution:

(a) We are given that the bacteria culture is growing at a rate proportional to its size, so we know it is growing exponentially. So, we are looking for an equation of the form $P(t) = P_0 e^{kt}$. We are looking for k.
We are told that $P(1) = \frac{3}{2}P_0$, so we use this to solve for k:

$$\frac{3}{2}P_0 = P_0 e^{k \cdot 1}$$

$$\frac{3}{2} = e^k$$

$$k = \boxed{\ln\left(\frac{3}{2}\right)}$$

(b) Plugging in, $P(t) = P_0 e^{\ln(3/2)t}$. Rewriting in simplified form: $\boxed{P(t) = P_0 \left(\frac{3}{2}\right)^t}$

(c) We are told $P(2) = 900$. We plug in to our equation from (b):

$$900 = P_0 \left(\frac{3}{2}\right)^2$$

$$900 = P_0 \left(\frac{9}{4}\right)$$

$$P_0 = \boxed{400 \text{ cells}}$$