Answer the following problems, showing all of your work and simplifying your solutions where possible unless otherwise stated.

No calculators, notes, books, electronic devices, internet access, AI tools etc. are allowed. This is a closed book, closed note exam.

1. (22 pts) Evaluate the integrals.

   (a) $\displaystyle\int \frac{2x^2}{\sqrt{4-x^2}}\mathrm{d}x$

   Solution:

   This is a good candidate for a trig sub. Let $x = 2\sin\theta$ (you could also use $2\cos\theta$). Then $\mathrm{d}x = 2\cos\theta\mathrm{d}\theta$.

$$\begin{aligned}
\int \frac{2x^2}{\sqrt{4-x^2}}\mathrm{d}x &= \int \frac{2(2\sin\theta)^2(2\cos\theta)}{\sqrt{4-4\sin^2\theta}}\mathrm{d}\theta \\
&= \int \frac{2\cdot 8\sin^2\theta\cos\theta}{2\sqrt{1-\sin^2\theta}}\mathrm{d}\theta \\
&= \int \frac{8\sin^2\theta\cos\theta}{\sqrt{\cos^2\theta}}\mathrm{d}\theta \\
&= 8\int \sin^2\theta\mathrm{d}\theta \\
&= 8\int \frac{1}{2}(1-\cos(2\theta))\mathrm{d}\theta \\
&= 4\theta - 2\sin(2\theta) + C
\end{aligned}$$

   We use the identity $\sin(2\theta) = 2\sin\theta\cos\theta$ and a reference triangle to find:

$$\begin{aligned}
4\theta - 2\sin(2\theta) + C &= 4\theta - 4\sin\theta\cos\theta + C \\
&= 4\arcsin(\frac{x}{2}) - 2x(\frac{\sqrt{4-x^2}}{2}) + C
\end{aligned}$$

   So

$$\int \frac{2x^2}{\sqrt{4-x^2}}\mathrm{d}x = 4\arcsin(\frac{x}{2}) - x\sqrt{4-x^2} + C$$

   (b) $\displaystyle\int_{-2}^{14} \frac{1}{\sqrt[4]{x+2}}\mathrm{d}x$

   Solution:

   Note that this is an improper integral of type 2. For $x = -2$, the integrand has a discontinuity, as the denominator evaluates to 0. Then we set up as:

$$\begin{aligned}\int_{-2}^{14} \frac{1}{\sqrt[4]{x+2}}\mathrm{d}x &= \lim_{t\to -2^+}\int_t^{14}\frac{1}{\sqrt[4]{x+2}}\mathrm{d}x\\ &= \lim_{t\to -2^+}\int_t^{14}(x+2)^{-1/4}\mathrm{d}x\\ &= \lim_{t\to -2^+}\frac{4}{3}(x+2)^{3/4}\Big|_t^{14}\\ &= \frac{4}{3}(16)^{3/4}-0\\ &= \frac{4}{3}(2^3)=\frac{32}{3}\end{aligned}$$

2. (46 pts) Let

$$f(x) = \ln(x).$$

(a) Find a Taylor series centered about $a = 1$ for $f(x)$.

(b) What is the radius and interval of convergence for the series in part a?

(c) Use $T_2(x)$ to estimate $f(1.1)$.

(d) Use Taylor's Remainder Formula to find an upper bound on the error in your estimation from part c.

Solution:

(a) To find the Taylor series, we must evaluate several derivative of $\ln(x)$ at 1 in order to establish a pattern.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln(x)$	0
1	$\frac{1}{x}$	1
2	$\frac{-1}{x^2}$	-1
3	$\frac{(-1)^2\cdot 2\cdot 1}{x^3}$	$(-1)^2\cdot 2!$
4	$\frac{(-1)^3(3\cdot 2\cdot 1)}{x^4}$	$(-1)^3\cdot 3!$
5	$\frac{(-1)^4\cdot 4!}{x^5}$	$(-1)^4\cdot 4!$

We can spot the pattern,

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n},\ n \geq 1$$

and $f^0(x) = 0$. Then

$$f^{(n)}(1) = (-1)^{n-1}(n-1)!$$

and

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$$

Note that the index starts at $n = 1$, since for $n = 1$, $f^{(n)}(1) = 0$. Therefore the first non-zero term occurs at $n = 1$.

(b) We find the radius and interval of convergence with the ratio test.

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(n+1)} \cdot \frac{n}{(x-1)^n} \right| \\ &= |x-1| \lim_{n\to\infty} \left| \frac{n+1}{n} \right| \\ L &= |x-1| \end{aligned}$$

Then

$$L < 1 \implies |x-1| < 1, 0 \leq x \leq 2$$

The radius of convergence is $R = 1$.

For the interval, we must test the convergence at the endpoints, $x = 0$ & $x = 2$. First, for $x = 0$:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(0-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n} = (-1)\sum_{n=1}^{\infty} \frac{1}{n}$$

This is the negative version of the harmonic series, which diverges.

For $x = 2$:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(2-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

This is the alternating harmonic series, which converges.

Therefore, the interval of convergence is $(0, 2]$.

(c) Using the Taylor series we have found, we have:

$$T_2(x) = (x-1) - \frac{1}{2}(x-1)^2$$

Then

$$T_2(1.1) = (1.1-1) - \frac{1}{2}(1.1-1)^2 = \frac{1}{10} - \frac{1}{2} \cdot \frac{1}{100} = \frac{19}{200}$$

(d) We know in general $R_2(x) = \dfrac{f^{(2+1)}(z)}{(2+1)!}(x-1)^{2+1}$.

From part 1,

$$|f^{(3)}(z)| = \left|\frac{2}{z^3}\right|, 1 \leq z \leq 1.1$$

To maximize this, we minimize the denominator, which occurs at $z = 1$. Then

$$|R_2(x)| \leq \frac{2}{3!}|1.1-1|^3 \leq \frac{1}{3} \cdot \frac{1}{1000}$$

Then our error bound is $|R_2(1.1)| \leq \frac{1}{3000}$.

As a side note, the actual error in this estimate is approximately $\frac{3.1}{10{,}000}$. The upper bound found here is relatively close to the true error.

3. (26 pts) The following questions are related.

(a) Find a Maclaurin series for $f(x) = \dfrac{1}{\sqrt{2\pi}}e^{-x^2}$.

(b) Find a Maclaurin series for $\displaystyle\int \frac{1}{\sqrt{2\pi}}e^{-x^2}\mathrm{d}x$. Include the radius of convergence.

(c) Use your answer in part a to evaluate $\displaystyle\int_0^{1/4} \frac{1}{\sqrt{2\pi}}e^{-x^2}dx$ exactly. (Hint: your solution will be in the form of a series).

Solution:

(a) This is a transformation of the Maclaurin series for e^x.

$$\frac{1}{\sqrt{2\pi}}e^{-x^2} = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}$$

(b) We can find the term-by-term integral.

$$\begin{aligned}\int \frac{1}{\sqrt{2\pi}}e^{-x^2}\mathrm{d}x &= \int \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!} \\ &= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int x^{2n}\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{x^{2n+1}}{(2n+1)} + C\end{aligned}$$

For the radius of convergence, $R = \infty$. We know this because the radius of the Maclaurin series of e^x is infinite, and none of the transformations- multiplication by a constant, squaring and taking negative x, and integrating- change the radius of convergence.

(c) This is evaluated with

$$\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{x^{2n+1}}{(2n+1)}\bigg|_0^{1/4}=\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{(1/4)^{2n+1}}{(2n+1)}-0=\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)4^{2n}}$$

4. (20 pts) Consider the following parametric equations:

$$x=-\ln(t) \quad y=-\frac{1}{2}t^2+1 \quad t>0$$

(a) Find the equation of the line tangent to the curve at the point $(0,\frac{1}{2})$.

(b) Eliminate the parameter t to find an equation for y in terms of x.

Solution:

(a) First, we find the t value that produces the point.

$$0=-\ln(t) \implies t=1$$

There is only one possible t value, so we must evaluate the slope of the tangent at $t=1$.

$$\frac{dy}{dt}=-t$$
$$\frac{dx}{dt}=-\frac{1}{t}$$
$$\frac{dy}{dx}=\frac{-t}{\frac{1}{-t}}$$
$$\frac{dy}{dx}=t^2$$

Then the slope of the tangent for $t=1$ is 1. Finally, using point/slope form:

$$y-\frac{1}{2}=1(x-0)$$

or

$$y=x+\frac{1}{2}$$

(b) To eliminate the parameter, we recognize:

$$x=-\ln(t) \implies t=e^{-x}$$

Then,

$$y=-\frac{1}{2}(e^{-x})^2+1=-\frac{1}{2}e^{-2x}+1$$

5. (36 points) Consider the polar equation $r^2 = \cos^2\theta$.

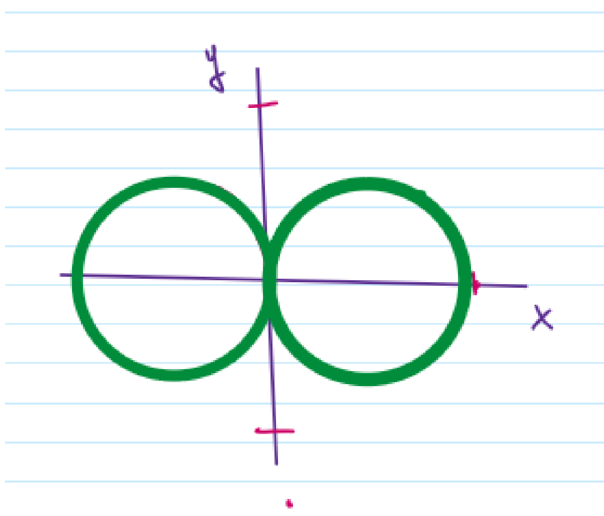
   (a) Sketch the polar curve.

   (b) **Evaluate an integral** to find the total area inside the curve.

   (c) **Evaluate an integral** to find the length of the curve.

Solution:

(a) This polar curve produces two circles with radius 1/2, centered at $(1/2, 0)$ and $(-1/2, 0)$:



(b) The area inside a polar curve is:

$$A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta$$

We are already given r^2. Additionally, though we could evaluate this from $\theta = 0$ to $\theta = 2\pi$, we note that there is symmetry: there are four half-circles, the first of which is drawn for $0 \le \theta \le \frac{\pi}{2}$. So a valid way to take advantage of this is:

$$A = 4\int_0^{\pi/2} \frac{1}{2}\cos^2(\theta)\mathrm{d}\theta$$

Evaluating the integral:

$$\begin{aligned} 4\int_0^{\pi/2} \frac{1}{2}\cos^2(\theta)\mathrm{d}\theta &= 4\int_0^{\pi/2} \frac{1}{4}(1+\cos(2\theta))\mathrm{d}\theta \\ &= \theta + \frac{1}{2}\sin(2\theta)\Big|_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2} \end{aligned}$$

Note that this makes sense, because there are two circles with radius 1/2, and geometrically, $A = 2 \cdot \pi(1/2)^2 = 2\pi/4 = \pi/2$.

6. Polar arclength is computed with the following integral:

$$L = \int_\alpha^\beta \sqrt{r^2 + (dr/d\theta)^2} \mathrm{d}\theta$$

We are given r^2. We must find $\mathrm{d}r/\mathrm{d}\theta$.

$$\begin{aligned}
\frac{d}{d\theta}\left[r^2 = \cos^2\theta\right] \implies 2r\frac{\mathrm{d}r}{\mathrm{d}\theta} &= 2\cos\theta(-\sin\theta) \\
\frac{\mathrm{d}r}{\mathrm{d}\theta} &= \frac{-\cos\theta\sin\theta}{r} \\
\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 &= \left(\frac{-\cos\theta\sin\theta}{r}\right)^2 \\
\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 &= \left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right) \\
\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 &= \left(\frac{\cos^2\theta\sin^2\theta}{\cos^2\theta}\right) \\
\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 &= \sin^2\theta
\end{aligned}$$

Then the integral is:

$$L = 4\int_0^{\pi/2} \sqrt{\cos^2\theta + \sin^2\theta} \mathrm{d}\theta = 4\theta\Big|_0^{\pi/2} = 2\pi$$

This makes sense, because geometrically the circumference of the two circles is

$$C = 2 \cdot (2\pi(1/2)) = 2\pi$$

Trigonometric Identities

$$\cos^2(x) = \frac{1}{2}(1+\cos(2x)) \quad \sin^2(x) = \frac{1}{2}(1-\cos(2x)) \quad \sin(2x) = 2\sin(x)\cos(x) \quad \cos(2x) = \cos^2(x)-\sin^2(x)$$

Inverse Trigonometric Integral Identities

$$\int \frac{\mathrm{d}u}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C,\ u^2 < a^2$$

$$\int \frac{\mathrm{d}u}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{\mathrm{d}u}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C,\ u^2 > a^2$$

Common Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots \qquad R = 1$$