Write your name below. This exam is worth 100 points and has 6 questions. On each problem, you must show all your work to receive credit on that problem. You are allowed to use one page of notes (one sided). You cannot collaborate on the exam or seek outside help, nor can you use the recorded lectures, a calculator, any computational software, or material you find online.

Name:

1. (21 points, 7 each) In each of the following problems, either show that the statement is always true or provide a counterexample that shows it is false.

   (a) The function
   $$p(\mathbf{x}) = x^2 - 2xz + y^2 - 2yz + 2z^2 - 2x - 2y - 2z + 4$$
   does not have a minimum value on $\mathbb{R}^3$.

   (b) If A^+ is the pseudoinverse of A then $(A^+A)^T = A^+A$.

   (c) The quadratic form $q(\mathbf{v}) = \mathbf{v}^T K\mathbf{v}$ defines a linear transformation $Q : \mathbb{R}^n \to \mathbb{R}$.

Solution: (a) True. We see that
$$p(\mathbf{x}) = \mathbf{x}^T K\mathbf{x} - 2\mathbf{x}^T\mathbf{f} + c$$
with
$$K = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix} \qquad \mathbf{f} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad c = 4$$
We see that K is positive semidefinite and $\mathbf{f} \notin \text{img}K$:
$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ -1 & -1 & 2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 2 \end{array}\right)$$
$$\to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right)$$
Since $p(\mathbf{x})$ only has a minimum if $\mathbf{f} \in \text{img}K$ when $K \geq 0$, we conclude it does not have a minimum.

(b) True. If we use the reduced SVD of A then we have
$$A^+A = Q\Sigma^{-1}P^T P\Sigma Q^T$$
Since P has orthonormal columns, $P^T P = I$ and we have
$$\begin{aligned} A^+A &= Q\Sigma^{-1}\Sigma Q^T \\ &= QQ^T \end{aligned}$$
and this is symmetric:
$$\begin{aligned} (QQ^T)^T &= (Q^T)^T Q^T \\ &= QQ^T \end{aligned}$$
so
$$(A^+A)^T = A^+A$$

(c) False. We see that $q(2\mathbf{v}) = (2\mathbf{v})^T K(2\mathbf{v}) = 4\mathbf{v}^T K\mathbf{v} = 4q(\mathbf{v}) \neq 2q(\mathbf{v})$

2. (20 points) Let A $= \begin{pmatrix} -1 & 0 & 3 \\ 3 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix}$

   (a) (10 points) Find the eigenvectors of A.

   (b) (3 points) Is A diagonalizable? Why or why not?

   (c) (7 points) If A is diagonalizable, find the matrices S and Λ that diagonalize it. If A is **not** diagonalizable, find the Jordan Canonical Form J and matrix S such that $A = SJS^{-1}$. You do not need to calculate S^{-1} in either case.

Solution: (a) We first need to find the eigenvalues from the characteristic polynomial:

$$\begin{aligned}
\det(A - \lambda I) &= \det \begin{pmatrix} -1-\lambda & 0 & 3 \\ 3 & 2-\lambda & -2 \\ 0 & 0 & 2-\lambda \end{pmatrix} \\
&= (2-\lambda)\det \begin{pmatrix} -1-\lambda & 0 \\ 3 & 2-\lambda \end{pmatrix} \\
&= (2-\lambda)(-1-\lambda)(2-\lambda) \\
&= -(2-\lambda)^2(1+\lambda)
\end{aligned}$$

The roots of this are our eigenvalues, so we have $\lambda = 2$ with algebraic multiplicity 2 and $\lambda = -1$ with algebraic multiplicity 1.

We find the eigenvectors by finding $\ker(A - \lambda I)$ for each λ:
$\lambda = -1$:

$$(A + I) = \begin{pmatrix} 0 & 0 & 3 \\ 3 & 3 & -2 \\ 0 & 0 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda = 2$:

$$(A - 2I) = \begin{pmatrix} -3 & 0 & 3 \\ 3 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

This is the only eigenvector for $\lambda = 2$, since $(A - 2I)$ only has one free variable.

(b) Since the eigenvalue $\lambda = 2$ only has one eigenvector, it's geometric multiplicity is $1 \neq 2$, so it is an incomplete eigenvalue. A is not a complete matrix and so is not diagonalizable.

(c) Our matrix is not diagonalizable, so we calculate the Jordan factorization. Our $\lambda = 2$ eigenvalue is the incomplete one, and we are missing one eigenvector. This means that we will have a Jordan chain starting with $\mathbf{v}_2$ that will be length 2.

Our Jordan canonical form of the matrix is therefore

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

To find the matrix S, we calculate the second vector in our Jordan chain:

$$(A - 2I)\mathbf{v}_3 = \mathbf{v}_2$$
$$\left(\begin{array}{ccc|c} -3 & 0 & 3 & 0 \\ 3 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
$$\to \left(\begin{array}{ccc|c} -3 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

So our S matrix, made up of all of the generalized eigenvectors, is

$$S = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. (20 points) Let

$$A = \begin{pmatrix} 1 & 2 \\ -3 & -1 \\ -1 & 3 \end{pmatrix}$$

(a) (4 points) Show that A has singular values $\sigma_1 = \sqrt{15}$ and $\sigma_2 = \sqrt{10}$.

(b) (7 points) Find the best rank 1 approximation of A.

(c) (9 points) Find the pseudoinverse of A.

Solution: (a) We find the eigenvalues of $A^T A$:

$$\begin{aligned} A^T A &= \begin{pmatrix} 1 & -3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -3 & -1 \\ -1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 11 & 2 \\ 2 & 14 \end{pmatrix} \\ \det(A - \lambda I) &= (11-\lambda)(14-\lambda) - 2^2 \\ &= 154 - 25\lambda + \lambda^2 - 4 \\ &= \lambda^2 - 25\lambda + 150 \\ &= (\lambda - 10)(\lambda - 15) \end{aligned}$$

So $A^T A$ has eigenvalues 15 and 10. The singular values of A are the square roots of these eigenvalues:

$$\begin{aligned} \sigma_1 &= \sqrt{15} \\ \sigma_2 &= \sqrt{10} \end{aligned}$$

(b) The best rank 1 approximation of A is given by

$$\tilde{A}_1 = \mathbf{p}_1 \sigma_1 \mathbf{q}_1^T$$

so we need the first singular vectors. Our $\mathbf{q}_1$ is the unit eigenvector of $A^T A$ with eigenvalue 15:

$$\begin{aligned} (A - 15I) &= \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \to \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \\ \mathbf{q}_1 &= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned}$$

We get $\mathbf{p}_1$ from our formula:

$$\begin{aligned} \mathbf{p}_1 = \frac{A\mathbf{q}_1}{\sigma_1} &= \frac{1}{\sqrt{15}} \begin{pmatrix} 1 & 2 \\ -3 & -1 \\ -1 & 3 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \frac{1}{5\sqrt{3}} \begin{pmatrix} 5 \\ -5 \\ 5 \end{pmatrix} \\ &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \end{aligned}$$

Substitution now gives

$$\begin{aligned}\tilde{A}_1 &= \mathbf{p}_1\sigma_1\mathbf{q}_1^T \\ &= \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\sqrt{15}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ -1 & -2 \\ 1 & 2 \end{pmatrix}\end{aligned}$$

(c) Our psuedoinverse is given by

$$A^+ = Q\Sigma^{-1}P^T$$

So we will also need $\mathbf{q}_2$ and $\mathbf{p}_2$. We find $\mathbf{q}_2$ by finding the eigenvector of A^TA with eigenvalue 10:

$$(A - 10I) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$
$$\mathbf{q}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

We get $\mathbf{p}_2$ from our formula:

$$\begin{aligned}\mathbf{p}_2 = \frac{A\mathbf{q}_2}{\sigma_2} &= \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 2 \\ -3 & -1 \\ -1 & 3 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix} \\ &= \frac{1}{5\sqrt{2}}\begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\end{aligned}$$

Now we have P and Q, and Σ is diagonal, so we see that

$$A^+ = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} 1/\sqrt{15} & 0 \\ 0 & 1/\sqrt{10} \end{pmatrix}\begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$
$$A^+ = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\frac{1}{\sqrt{30}}\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{pmatrix}\frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} & \sqrt{3} \end{pmatrix}$$
$$A^+ = \frac{1}{30}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & -2 & 2 \\ 0 & 3 & 3 \end{pmatrix}$$
$$A^+ = \frac{1}{30}\begin{pmatrix} 2 & -8 & -4 \\ 4 & -1 & 7 \end{pmatrix}$$

4. (20 points) Let L be the linear function $L : \mathbb{R}^3 \to \mathbb{R}^2$ that is given in the standard basis by

$$L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ y + 2z \end{pmatrix}$$

For all of the questions below, we wish to find the bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ that put L into the canonical form.

(a) (2 points) Find the matrix representation of L in the standard basis.

(b) (8 points) What basis should we choose for $\mathbb{R}^3$?

(c) (6 points) What basis should we choose for $\mathbb{R}^2$?

(d) (4 points) Verify that your bases are the correct bases.

Solution: (a) The matrix representation in the standard basis is given by

$$\begin{aligned}(L(\mathbf{e}_1)\ L(\mathbf{e}_2)\ L(\mathbf{e}_3)) &= \left(L\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\ L\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\ L\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}\end{aligned}$$

(b) For the $\mathbb{R}^3$ basis, we need a basis for the coimage. The matrix is rank 2, so we can take the columns of A^T for our basis:

$$A^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{pmatrix} \qquad \text{coimg} A = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

We then add a basis vector for the kernel of A, as that is the orthogonal complement we need:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\ker A = \text{span} \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

Collecting these together gives our set of basis vectors we need for $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

(c) For the $\mathbb{R}^2$ basis, we calculate function's values for the $\mathbb{R}^3$ basis vectors:

$$\begin{aligned}\mathbf{t}_1 &= L(\mathbf{s}_1) = A\mathbf{s}_1 \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 14 \\ 8 \end{pmatrix} \\ \mathbf{t}_2 &= L(\mathbf{s}_2) = A\mathbf{s}_2 \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 8 \\ 5 \end{pmatrix}\end{aligned}$$

So our basis for $\mathbb{R}^2$ is

$$\left\{ \begin{pmatrix} 14 \\ 8 \end{pmatrix}, \begin{pmatrix} 8 \\ 5 \end{pmatrix} \right\}$$

(d) To vefrify, we need to show that

$$T^{-1}AS = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

where

$$S = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} 14 & 8 \\ 8 & 5 \end{pmatrix}$$

Find T^{-1}:

$$\begin{aligned}T^{-1} &= \frac{1}{70-64}\begin{pmatrix} 5 & -8 \\ -8 & 14 \end{pmatrix} \\ &= \frac{1}{6}\begin{pmatrix} 5 & -8 \\ -8 & 14 \end{pmatrix}\end{aligned}$$

And calculate

$$\begin{aligned}T^{-1}AS &= \frac{1}{6}\begin{pmatrix} 5 & -8 \\ -8 & 14 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \\ &= \frac{1}{6}\begin{pmatrix} 5 & -8 \\ -8 & 14 \end{pmatrix}\begin{pmatrix} 14 & 8 & 0 \\ 8 & 5 & 0 \end{pmatrix} \\ &= \frac{1}{6}\begin{pmatrix} 70-64 & 40-40 & 0 \\ -112+112 & -64+70 & 0 \end{pmatrix} \\ &= \frac{1}{6}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\end{aligned}$$

5. (10 points) Let A be a symmetric 2×2 matrix with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ and eigenvalue λ_1 has the eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

   (a) (5 points) Find A.

   (b) (5 points) Find the matrix exponential e^{At}.

Solution: (a) We find A from it's diagonalization, $A = S\Lambda S^{-1}$, but first we must find the second eigenvector. Since A is symmetric we know it's eigenvectors with distinct eigenvalues are orthogonal, so we are looking for a vector orthogonal to $\mathbf{v}_1$:

$$\mathbf{v}_1^T \mathbf{v}_2 = 0$$

So we are solving the homogeneous equation with the matrix

$$\begin{pmatrix} 1 & -2 \end{pmatrix}$$

and so we have

$$\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Now we can diagonalize:

$$\begin{aligned}
A &= \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}^{-1} \\
A &= \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{(-4-1)} \begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix} \\
A &= \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \\
A &= \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix} \\
A &= \frac{1}{5} \begin{pmatrix} -3 & -4 \\ -4 & 3 \end{pmatrix}
\end{aligned}$$

(b) Since we already have the diagonalization of A, the matrix exponential is given by

$$\begin{aligned}
e^{At} &= e^{S\Lambda t S^{-1}} = S e^{\Lambda t} S^{-1} \\
&= \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2e^{-t} & e^{-t} \\ e^{t} & -2e^{t} \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} 4e^{-t} + e^{t} & 2e^{-t} - 2e^{t} \\ 2e^{-t} - 2e^{t} & e^{-t} + 4e^{t} \end{pmatrix}
\end{aligned}$$

6. (9 points) Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 3 \\ 3 \\ 0 \\ 0 \end{pmatrix}$$

Find the vector in img$A$ that is closest to $\mathbf{b}$ with the standard dot product.

Solution: We solve the normal equations to find the coordinates of our vector:

$$A^T A\mathbf{x}^* = A^T\mathbf{b}$$

to find our least squares solution.

$$A^T A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}$$

$$A^T\mathbf{b} = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \end{pmatrix}$$

Solving our system:

$$\left(\begin{array}{cc|c} 6 & 4 & 9 \\ 4 & 6 & 9 \end{array}\right) \xrightarrow[R_2 \to R_2 - \frac{2}{3}R_1]{} \left(\begin{array}{cc|c} 6 & 4 & 9 \\ 0 & \frac{10}{3} & 3 \end{array}\right)$$

$$\xrightarrow[R_2 \to 3R_2]{} \left(\begin{array}{cc|c} 6 & 4 & 9 \\ 0 & 10 & 9 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - \frac{2}{5}R_2} \left(\begin{array}{cc|c} 6 & 0 & \frac{27}{5} \\ 0 & 10 & 9 \end{array}\right)$$

$$\xrightarrow[R_2 \to \frac{1}{10}R_2]{R_1 \to \frac{1}{6}R_1} \left(\begin{array}{cc|c} 1 & 0 & 27/30 \\ 0 & 1 & 9/10 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 0 & 9/10 \\ 0 & 1 & 9/10 \end{array}\right)$$

so our least squares solution is

$$\mathbf{x}^* = \frac{9}{10}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Finally, to find the closest point in img$A$, we find $A\mathbf{x}^*$:

$$A\mathbf{x}^* = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{9}{10}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{9}{10}\begin{pmatrix} 3 \\ 3 \\ -1 \\ 1 \end{pmatrix}$$